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# Symmetrised powers of rotation group representations

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**Abstract.** The results given in this paper permit the unambiguous evaluation of all possible Kronecker products of the irreducible representations (tensor and spinor) of $O_n$ and $SO_n$ for $n = 2\nu$ and $n = 2\nu + 1$. A complete resolution of the second and third powers of the basic spinor representations of $SO_{2\nu}$ and $SO_{2\nu+1}$ is given, together with a prescription for analysing the fourth power of these representations. Detailed application is made to the enumeration of properties of $SO_{10}$ relevant to grand unified theories, and sufficient information given to resolve the fourth power of any representation of $SO_{10}$.

## 1. Introduction

The $n$-dimensional rotation groups play an important role in many areas of physics and chemistry. They arise, for example, in the description of symmetrised orbitals in quantum chemistry (Wybourne 1973), in fermion many-body theory (Fukutome *et al* 1977), in boson models of nuclei (Arima and Iachello 1976), grand unified theories (Gell-Mann *et al* 1978) and in supergravity theories (Cremmer and Julia 1979). In all these applications the analysis of the Kronecker product of irreducible representations (irreps) of the $n$-dimensional rotation group is of significance.

Interest in the rotation groups has greatly increased in recent times with study of candidate groups for grand unified theories of the weak, electromagnetic and strong interactions. The group $SO_{10}$ appears to be of particular significance (Fritzch and Minkowski 1975, Chanowitz *et al* 1979, Buras *et al* 1978, Georgi and Nanopoulos 1979, Witten 1979). In these cases the fermions are usually associated with the spinor irreps of some $SO_n$ and the bosons with ordinary irreps of the same $SO_n$. All of these irreps may be generated from the basic spinor irreps of $SO_n$. Renormalisability constraints usually limit interest to Kronecker powers of at most fourth order in any particular irrep of $SO_n$.

A definitive study of the analysis of the spinor irreps of $SO_n$ was made by Brauer and Weyl (1935). These authors gave a complete description of the resolutions of the Kronecker square of the basic spin irrep for both $n = 2\nu$ and $n = 2\nu + 1$. This analysis was extended by Murnaghan (1938), who introduced the use of difference characters in resolving the Kronecker square of the irreps of $SO_{2\nu}$. Further results were obtained by Littlewood (1947, 1948, 1950). Littlewood was able to exploit known isomorphisms and automorphisms to resolve all powers of the basic spin irreps of the rotation groups in three to eight dimensions. He noted in his 1947 paper 'The construction of the

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concomitants of degree higher than 2 in 10 or more variables would appear to present a formidable problem'.

Butler and Wybourne (1969) showed how it was possible to exploit Littlewood's results to reduce Kronecker products for the full orthogonal groups $O_n$ and the rotation groups $SO_n$. Furthermore, they gave a prescription for analysing the Kronecker square of arbitrary irreps of $O_n$ and $SO_n$.

A number of their results on Kronecker products have been simplified and extended elsewhere (King 1975a, b), and this work is continued here. Explicit formulae are given for a complete set of fundamental products from which all possible products of irreps of $O_n$ and $SO_n$ may be evaluated both for $n=2\nu$ and for $n=2\nu+1$.

The explicit resolution of the basic Kronecker squares into their symmetric and antisymmetric parts is then given, followed by a complete resolution of the Kronecker cubes of the basic spin irreps of $SO_{2\nu+1}$ and $SO_{2\nu}$, together with a prescription for analysing explicitly the Kronecker fourth powers of these irreps. These results permit analysis of the Kronecker second, third and fourth powers of any irrep (spinor or tensor) of the groups $SO_{2\nu+1}$ or $SO_{2\nu}$ to be made unambiguously. Detailed application of these results to the enumeration of properties of $SO_{10}$ relevant to grand unified theories is made, and sufficient information given to resolve any fourth power of any irrep of $SO_{10}$.

## 2. Schur-function series

Throughout this paper we shall make extensive use of the theory of Schur functions (cf Littlewood 1950, Macdonald 1979, Wybourne 1970). The following $S$-function series (King 1975b) play a key role:

$$A=\sum_{\alpha}(-1)^{a/2}\{\alpha\},\qquad B=\sum_{\beta}\{\beta\},\qquad C=\sum_{\gamma}(-1)^{c/2}\{\gamma\},$$

$$D=\sum_{\delta}\{\delta\},\qquad E=\sum_{\varepsilon}(-1)^{(e+r)/2}\{\varepsilon\},\qquad F=\sum_{\zeta}\{\zeta\},$$

$$G=\sum_{\varepsilon}(-1)^{(e-r)/2}\{\varepsilon\},\qquad H=\sum_{\zeta}(-1)^{z}\{\zeta\},\qquad L=\sum_{m}(-1)^{m}\{1^{m}\},$$

$$M=\sum_{m}\{m\},\qquad P=\sum_{m}(-1)^{m}\{m\},\qquad Q=\sum_{m}\{1^{m}\},$$

where $(\alpha)$ and $(\gamma)$ are mutually conjugate partitions, which in the Frobenius notation (cf Littlewood 1950) take the form

$$(\alpha)=\begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ a_1+1 & a_2+1 & \ldots & a_r+1\end{pmatrix},\qquad (\gamma)=\begin{pmatrix} c_1+1 & c_2+1 & \ldots & c_r+1 \\ c_1 & c_2 & \ldots & c_r\end{pmatrix}.$$

$(\delta)$ is a partition into even parts only and $(\beta)$ is conjugate to $(\delta)$. $(\zeta)$ is any partition and $(\varepsilon)$ is any self-conjugate partition. $(\alpha)$, $(\gamma)$, $(\varepsilon)$ and $(\zeta)$ are partitions of $a$, $c$, $e$, and $z$ respectively, whilst $r$ is the Frobenius rank of $(\alpha)$, $(\gamma)$ and $(\varepsilon)$.

These series occur as mutually inverse pairs:

$$AB=CD=EF=GH=LM=PQ=\{0\}=1. \tag{2.1}$$

Also

$$LA = PC = E, \qquad MB = QD = F,$$
$$MC = AQ = G, \qquad LD = PB = H. \tag{2.2}$$

In addition to the above series we shall make use of both

$$R = \{0\} - 2\sum_{a,b} (-1)^{a+b+1}\begin{Bmatrix} a \\ b \end{Bmatrix}, \qquad S = \{0\} + 2\sum_{a,b} \begin{Bmatrix} a \\ b \end{Bmatrix},$$

where the Frobenius notation has been used once again, and

$$V = \sum_{\omega} (-1)^q \{\tilde{\omega}\}, \qquad W = \sum_{\omega} (-1)^q \{\omega\},$$

$$X = \sum_{\omega} \{\tilde{\omega}\}, \qquad Y = \sum_{\omega} \{\omega\},$$

where $(\omega)$ is a partition of an even number into at most two parts, the second of which is $q$, and $\tilde{\omega}$ is the conjugate of $\omega$.

We readily find that

$$RS = VW = \{0\} = 1, \tag{2.3}$$

and

$$PM = AD = W, \qquad LQ = BC = V,$$
$$MQ = FG = S, \qquad LP = HE = R. \tag{2.4}$$

We make frequent use of $S$-function division, signified by /, which is governed by the Littlewood–Richardson rule (Littlewood 1950, p 94). In the case of division of $S$-function series we shall often write, for example,

$$\{\lambda/A\} = \sum_{\alpha} (-1)^{a/2} \{\lambda/\alpha\}$$

and

$$\{\lambda/B\} = \sum_{\beta} \{\lambda/\beta\}.$$

The list of remarkable identities involving $S$-function multiplication and division includes the following:

$$\{\sigma \cdot \tau\}/Z = \{\sigma/Z\} \cdot \{\tau/Z\} \qquad \text{for } Z = L, M, P, Q, R, S, V \text{ and } W, \tag{2.5}$$

$$\{\sigma \cdot \tau\}/Z = \sum_{\zeta} \{\sigma/\zeta Z\} \cdot \{\tau/\zeta Z\} \qquad \text{for } Z = B, D, F \text{ and } H \tag{2.6}$$

and

$$\{\sigma \cdot \tau\}/Z = \sum_{\zeta} (-1)^z \{\sigma/\zeta Z\} \cdot \{\tau/\tilde{\zeta} Z\} \qquad \text{for } Z = A, C, E \text{ and } G \tag{2.7}$$

where · signifies $S$-function multiplication, which is governed by the same Littlewood–Richardson rule:

$$\{\mu\} \cdot \{\nu\} = \sum_{\lambda} m^{\lambda}_{\mu\nu} \{\lambda\}, \qquad \{\lambda/\mu\} = \sum_{\nu} m^{\lambda}_{\mu\nu} \{\nu\}. \tag{2.8}$$

An additional notational development, which we use a great deal, arises most naturally through the application of the Littlewood–Richardson rule to give

$$\{1^s\}\cdot\{\rho\}=\sum_t \{1^{s+t};\rho/1^t\}, \tag{2.9}$$

where for any partition $\lambda$ into $p$ non-vanishing parts

$$\{1^r;\lambda\}=\{\lambda_1+1,\lambda_2+1,\ldots,\lambda_p+1,1^{r-p}\} \qquad \text{for } r\geqslant p. \tag{2.10}$$

The Young diagram of $(1^r;\lambda)$ is thus formed by adjoining the single column of length $r$ corresponding to $(1^r)$ to the left-hand side of the Young diagram of $\lambda$. In terms of the conjugacy symbol $\tilde{}$

$$\{1^r,\lambda\}=\widetilde{\{r,\tilde{\lambda}\}}. \tag{2.11}$$

This same symbol is given a meaning in the case $r<p$ by means of the modification rule (King 1971)

$$\{1^r;\lambda\}=(-1)^x\{1^{p-1};\lambda-h\} \qquad \text{with } h=p-r-1, \tag{2.12}$$

where $p$ is the number of parts of $\lambda$, and $h$ is the length of the continuous boundary strip or hook removed from the Young diagram of $\lambda$, starting from the foot of the first, or left-most, column and ending in the $x$th column. The corresponding term vanishes unless the resulting Young diagram signified by $\lambda-h$ is regular.

## 3. Orthogonal groups $O_n$

The orthogonal group $O_n$ has both *true* representations, commonly referred to as ordinary or *tensor* representations, and *projective* representations, commonly referred to as double-valued or *spinor* representations. The tensor irreps of $O_n$ are labelled by partitions $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_p)$ which serve to specify the corresponding characters $[\lambda]$. We denote (King 1975b) the characters of the spinor irreps of $O_n$ by $[\Delta;\lambda]$, where $\lambda$ is a partition associated with a set of tensor indices and $\Delta$ is a symbol associated with a spinor index. The notation for the character of the basic spin irrep will frequently be contracted to just $\Delta=[\Delta;0]$. This irrep of $O_n$ is of dimension $2^\nu$ for both $n=2\nu$ and $n=2\nu+1$.

The identity representation of $O_n$ is the one-dimensional irrep whose character $[0]$ is unity for every element $A$ of the group. The alternating tensor is not an absolute invariant but rather a relative invariant under $O_n$. Correspondingly, there exists another one-dimensional irrep whose character $[0]^*$ is $\det A=\pm1$ for each element $A$ of $O_n$. For each irrep of $O_n$ there thus exists an associated irrep obtained by multiplying each matrix representing the group element $A$ by $\det A$. Thus if $[\lambda]$ and $[\Delta;\lambda]$ are characters of $O_n$, then

$$[\lambda]^*=[\lambda][0]^* \qquad \text{and} \qquad [\Delta;\lambda]^*=[\Delta;\lambda][0]^* \tag{3.1}$$

are also characters. The *associated characters* $[\lambda]^*$ and $[\Delta;\lambda]^*$ will differ from $[\lambda]$ and $[\Delta;\lambda]$ respectively unless the characters $[\lambda]$ and $[\Delta;\lambda]$ are zero for all elements of $O_n$ for which $\det A=-1$.

The following equivalence relations or modification rules hold for $O_n$ (King 1971, 1975b):

$$[\lambda]=(-1)^{x-1}[\lambda-h]^* \qquad \text{with } h=2p-n, \tag{3.2a}$$

$$[\Delta; \lambda] = (-1)^x [\Delta; \lambda - h]^* \qquad \text{with } h = 2p - n - 1, \tag{3.2b}$$

where the symbols $\lambda$, $p$, $h$, $x$ and $(\lambda - h)$ have exactly the same interpretation as in (2.12). Once again the corresponding irrep vanishes unless the resulting Young diagram, signified by $\lambda - h$, is regular. These equivalences allow us to restrict ourselves to partitions of at most $\nu$ parts for both $n = 2\nu$ and $n = 2\nu + 1$, and to establish that the inequivalent irreps of $O_n$ are those specified by the following characters:

$$\begin{array}{lll} O_{2\nu} & [\lambda], [\lambda]^* & \text{for } p < \nu, \\ & [\lambda] & \text{for } p = \nu, \\ & [\Delta; \lambda] & \text{for } p \leq \nu, \end{array} \tag{3.3a}$$

and

$$\begin{array}{lll} O_{2\nu+1} & [\lambda], [\lambda]^* & \text{for } p \leq \nu, \\ & [\Delta; \lambda], [\Delta; \lambda]^* & \text{for } p \leq \nu. \end{array} \tag{3.3b}$$

All the irreps $O_n$ are self-contragredient, whilst just the irreps of $O_{2\nu}$ with characters $[\lambda]$ with $p = \nu$ or $[\Delta; \lambda]$ with $p \leq \nu$ are self-associate.

The connection between the labels (3.3) and the usual highest-weight labels is such that the irreps with characters $[\lambda]$ and $[\Delta; \lambda]$ have highest-weight vectors $(\lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_\nu)$ and $(\Delta; \lambda) = (\lambda_1 + \frac{1}{2}, \lambda_2 + \frac{1}{2}, \ldots, \lambda_\nu + \frac{1}{2})$ respectively, where the notation for the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ has been extended in such a way that $\lambda_q = 0$ for $q = p+1, p+2, \ldots, \nu$. It is sometimes convenient to call the sum of the components of the highest-weight vector of an irrep the *rank* of the irrep or corresponding character, so that if $\lambda$ is a partition of $l$ the ranks of $[\lambda]$ and $[\Delta; \lambda]$ are simply $l$ and $l + \frac{1}{2}\nu$.

The spin characters $[\Delta; \lambda]$ of $O_n$ may be written as a product of the basic spin character $\Delta$ and tensor characters by noting (King 1975b) that for $O_{2\nu}$

$$[\Delta; \lambda] = \Delta[\lambda/P] \tag{3.4a}$$

and inversely

$$\Delta[\lambda] = [\Delta; \lambda/Q], \tag{3.4b}$$

whilst for $O_{2\nu+1}$

$$[\Delta; \lambda] = \Delta[\lambda/P^*] \tag{3.5a}$$

and

$$\Delta[\lambda] = [\Delta; \lambda/Q^*], \tag{3.5b}$$

where it has been convenient to define

$$P^* = \sum_m (-1)^m \{m\}^{(*)m} \tag{3.6a}$$

and

$$Q^* = \sum_m \{1^m\}^{(*)m} \tag{3.6b}$$

and it is to be understood that, for example,

$$[\lambda/m^{(*)m}] = [\lambda/m]^{(*)m} = \begin{cases} [\lambda/m] & \text{if } m \text{ is even,} \\ [\lambda/m]^* & \text{if } m \text{ is odd,} \end{cases} \tag{3.7a}$$

since of course

$$([0]^*)^m = [0]^{(*)^m} = \begin{cases} [0] & \text{if } m \text{ is even,} \\ [0]^* & \text{if } m \text{ is odd.} \end{cases} \tag{3.7b}$$

## 4. Rotation groups $SO_n$

In the case of the groups $O_{2\nu+1}$ each irrep remains irreducible on restriction to the unimodular subgroup $SO_{2\nu+1}$. Moreover, the distinction between associate representations is lost, so that under this restriction

$$O_{2\nu+1} \downarrow SO_{2\nu+1} \qquad [\lambda] \downarrow [\lambda], [\lambda]^* \downarrow [\lambda], [\Delta; \lambda] \downarrow [\Delta; \lambda], [\Delta; \lambda]^* \downarrow [\Delta; \lambda]. \tag{4.1}$$

The inequivalent irreps of $SO_{2\nu+1}$ have characters

$$SO_{2\nu+1} \qquad [\lambda], [\Delta; \lambda], \qquad p \leqslant \nu. \tag{4.2}$$

In the case of the groups $O_{2\nu}$ only those irreps which are *not* self-associate remain irreducible on restriction to $SO_{2\nu}$. Each self-associate irrep with character either $[\lambda]$ with $sp = \nu$ or $[\Delta; \lambda]$ with $p \leqslant \nu$ reduces to a sum of two inequivalent irreps of $SO_{2\nu}$ under this restriction, so that

$$\begin{aligned} O_{2\nu} \downarrow SO_{2\nu} \qquad & [\lambda] \downarrow [\lambda], [\lambda]^* \downarrow [\lambda], && p < \nu, \\ & [\lambda] \downarrow [\lambda]_+ + [\lambda]_-, && p = \nu, \\ & [\Delta; \lambda] \downarrow [\Delta; \lambda]_+ + [\Delta; \lambda]_-, && p \leqslant \nu. \end{aligned} \tag{4.3}$$

The inequivalent irreps of $SO_{2\nu}$ have characters

$$\begin{aligned} SO_{2\nu} \qquad & [\lambda], && p < \nu, \\ & [\lambda]_+, [\lambda]_-, && p = \nu, \\ & [\Delta; \lambda]_+, [\Delta; \lambda]_-, && p \leqslant \nu. \end{aligned} \tag{4.4}$$

Although the irreps with characters $[\lambda]_+$ and $[\lambda]_-$ and those with characters $[\Delta; \lambda]_+$ and $[\Delta; \lambda]_-$ are not equivalent, they are conjugate to one another under an involutory outer automorphism, $^\dagger$, of $SO_{2\nu}$ involving a matric determinant $-1$. Under this automorphism

$$[\lambda]^\dagger = [\lambda], \qquad p < \nu, \tag{4.5a}$$

$$[\lambda]_\pm^\dagger = [\lambda]_\mp, \qquad p = \nu, \tag{4.5b}$$

$$[\Delta; \lambda]_\pm^\dagger = [\Delta; \lambda]_\mp, \qquad p \leqslant \nu. \tag{4.5c}$$

It follows that the dimensions of the irreps of $SO_{2\nu}$ with characters $[\lambda]_+$ and $[\lambda]_-$ are each equal to half that of the irrep of $O_{2\nu}$ with character $[\lambda]$ for $p = \nu$, whilst the dimensions of the irreps of $SO_{2\nu}$ with characters $[\Delta; \lambda]_+$ and $[\Delta; \lambda]_-$ are equal to half that of the irrep of $O_{2\nu}$ with character $[\Delta; \lambda]$ for $p \leqslant \nu$.

The highest weights of the irreps with characters $[\lambda]$ for $p < \nu$, $[\lambda]_\pm$ for $p = \nu$ and $[\Delta; \lambda]_\pm$ for $p \leqslant \nu$ are given by $(\lambda_1, \lambda_2, \ldots, \lambda_{\nu-1}, 0)$, $(\lambda_1, \lambda_2, \ldots, \lambda_{\nu-1}, \pm\lambda_\nu)$ and $(\lambda_1 + \frac{1}{2}, \lambda_2 + \frac{1}{2}, \ldots, \lambda_{\nu-1} + \frac{1}{2}, \pm\lambda_\nu \pm \frac{1}{2})$ respectively. Correspondingly, if $\lambda$ is a partition of $l$, the ranks of the characters $[\lambda]$ for $p < \nu$, $[\lambda]_+$, $[\lambda]_-$ for $p = \nu$ and $[\Delta; \lambda]_+$, $[\Delta; \lambda]_-$ for $p \leqslant \nu$ are $l$, $l$, $l - 2\lambda_\nu$, $l + \frac{1}{2}\nu$ and $l + \frac{1}{2}\nu - 2(\lambda_\nu + \frac{1}{2})$ respectively.

In the case of inequivalent, but conjugate, pairs of $SO_{2\nu}$ irreps it is convenient to denote the characters of their sums by

$$[\lambda]=[\lambda]_{+}+[\lambda]_{-}, \tag{4.6a}$$

$$[\Delta;\lambda]=[\Delta;\lambda]_{+}+[\Delta;\lambda]_{-}, \tag{4.6b}$$

and to introduce the difference characters (Murnaghan 1938, p290, Littlewood 1950, p246)

$$[\lambda]''=[\lambda]_{+}-[\lambda]_{-}, \tag{4.6c}$$

$$[\Delta;\lambda]''=[\Delta;\lambda]_{+}-[\Delta;\lambda]_{-}, \tag{4.6d}$$

so that

$$[\lambda]_{\pm}=\tfrac{1}{2}([\lambda]\pm[\lambda]''), \tag{4.7a}$$

$$[\Delta;\lambda]_{\pm}=\tfrac{1}{2}([\Delta;\lambda]\pm[\Delta;\lambda]''). \tag{4.7b}$$

The spinor characters and difference characters may be written as products of basic spin characters and difference characters with true characters and difference characters. Using the results due to Littlewood (1950) and the various $S$-functions series identities (2.1) and (2.2), it is easy to see that for $SO_{2\nu}$

$$[\Delta;\lambda]=\Delta[\lambda/P], \tag{4.8a}$$

$$[\Delta;\lambda]''=\Delta''[\lambda/M], \tag{4.8b}$$

and inversely

$$\Delta[\lambda]=[\Delta;\lambda/Q], \tag{4.9a}$$

$$\Delta''[\lambda]=[\Delta;\lambda/L]''. \tag{4.9b}$$

It then follows from (4.7) that

$$[\Delta;\lambda]_{\pm}=\sum_{m}(-1)^{m}\,\Delta_{\pm(-)^{m}}[\lambda/m], \tag{4.10a}$$

$$\Delta_{\pm}[\lambda]=\sum_{m}[\Delta;\lambda/1^{m}]_{\pm(-)^{m}}. \tag{4.10b}$$

In the case of tensor characters and difference characters it is convenient to write, as in (2.10), (El Samra and King 1979)

$$[\lambda_1+1,\lambda_2+1,\ldots,\lambda_\nu+1]=[\Box;\lambda], \tag{4.11a}$$

$$[\lambda_1+1,\lambda_2+1,\ldots,\lambda_\nu+1]''=[\Box;\lambda]'', \tag{4.11b}$$

$$[\Box;\lambda]_{\pm}=\tfrac{1}{2}([\Box;\lambda]\pm[\Box;\lambda]''), \tag{4.11c}$$

and

$$\Box=[\Box;0]=[1^{\nu}]=[1^{\nu}]_{+}+[1^{\nu}]_{-}, \tag{4.12a}$$

$$\Box''=[\Box;0]''=[1^{\nu}]''=[1^{\nu}]_{+}-[1^{\nu}]_{-}, \tag{4.12b}$$

$$\Box_{\pm}=[\Box;0]_{\pm}=[1^{\nu}]_{\pm}=\tfrac{1}{2}(\Box\pm\Box''). \tag{4.12c}$$

With this notation the analogues of the results (4.8) and (4.9) are

$$[\Box: \lambda] = \Box[\lambda/Y] + 2\sum_{s,t}(-1)^t[1^{\nu-1-t}][\lambda/(1+t+s)/s], \quad (4.13a)$$

$$[\Box; \lambda]'' = \Box''[\lambda/W], \quad (4.13b)$$

and inversely

$$\Box[\lambda] = [\Box; \lambda/X] + 2\sum_{s,t}[1^{\nu-1-t}; \lambda/1^{1+t+s}/1^s], \quad (4.14a)$$

$$\Box''[\lambda] = [\Box; \lambda/V]'', \quad (4.14b)$$

so that

$$[\Box; \lambda]_{\pm} = \sum_{\omega}\Box_{\pm(-)^q}[\lambda/\omega] + \sum_{s,t}(-1)^t[1^{\nu-1-t}][\lambda/(1+t+s)/s], \quad (4.15a)$$

$$\Box_{\pm}[\lambda] = \sum_{\omega}[\Box; \lambda/\tilde{\omega}]_{\pm(-)^q} + \sum_{s,t}[1^{\nu-1-t}; \lambda/1^{1+t+s}/1^s], \quad (4.15b)$$

where use has been made of the notation of (2.10).

These results may all be derived from the work of Littlewood (1950) through the use of $S$-function series identities (2.1)–(2.4), and the rules for evaluating Kronecker products of tensor irreps of $O_n$ (Littlewood 1958). These rules, taken in conjunction with the modification rules (3.2$a$), lead inexorably to the rather unwelcome double summations over $s$ and $t$.

To conclude this section, it is worth pointing out that for characters of $SO_{2\nu}$ the complete set of modification rules is

$$[\lambda] = (-1)^{x-1}[\lambda - h] \quad \text{with } h = 2p - 2\nu, \quad (4.16a)$$

$$[\Delta; \lambda] = (-1)^x[\Delta; \lambda - h] \quad \text{with } h = 2p - 2\nu - 1, \quad (4.16b)$$

$$[\Delta; \lambda]'' = (-1)^{x-1}[\Delta; \lambda - h]'' \quad \text{with } h = 2p - 2\nu - 1, \quad (4.16c)$$

$$[\Delta; \lambda]_{\pm} = (-1)^x[\Delta; \lambda - h]_{\mp} \quad \text{with } h = 2p - 2\nu - 1, \quad (4.16d)$$

$$[\Box; \lambda] = (-1)^{x-1}[\Box; \lambda - h] \quad \text{with } h = 2p - 2\nu - 2, \quad (4.16e)$$

$$[\Box; \lambda]'' = (-1)^x[\Box; \lambda - h]'' \quad \text{with } h = 2p - 2\nu - 2, \quad (4.16f)$$

$$[\Box; \lambda]_{\pm} = (-1)^{x-1}[\Box; \lambda - h]_{\mp} \quad \text{with } h = 2p - 2\nu - 2. \quad (4.16g)$$

The key modification rules appropriate to spinor and difference characters follow most readily from the fact that explicit determinantal forms for these characters allow $[\Delta; \lambda]$, $[\Delta; \lambda]''$, and $[\Box; \lambda]''$ to be written as products of $\Delta$, $\Delta''$, and $\Box''$ respectively, with formal irreducible characters (El-Samra and King 1979), whose modification rules are known (King 1971, 1975b).

## 5. Basic Kronecker products

The Kronecker squares of the basic spin irreps of $O_n$ were evaluated by Brauer and Weyl (1935), and may be conveniently written in the form

$$O_{2\nu} \qquad \Delta^2 = \sum_{r=0}^{2\nu}[1^r] = [1^\nu] + \sum_s([1^{\nu-1-s}] + [1^{\nu-1-s}]^*), \quad (5.1a)$$

$$O_{2\nu+1} \qquad \Delta^2 = \sum_{r=0}^{\nu} [1^{2r}] = \sum_s [1^{\nu-s}]^{(*)^{\nu-s}}, \tag{5.1b}$$

where use has been made of the equivalence $[1^r] = [1^{n-r}]^*$ which follows from (3.2*a*).

In the case of the groups $SO_{2\nu}$, (5.1*a*) yields

$$\Delta^2 = [1^\nu] + 2\sum_s [1^{\nu-1-s}], \tag{5.2a}$$

whilst for the difference characters (Butler and Wybourne 1969) we have

$$\Delta''^2 = [1^\nu] - 2\sum_s (-1)^s [1^{\nu-1-s}] \tag{5.2b}$$

and

$$\Delta\Delta'' = \square'' = [1^\nu]''. \tag{5.2c}$$

From these results it is easy to rederive those of Brauer and Weyl (1935),

$$\Delta_\pm \Delta_\pm = [1^\nu]_\pm + \sum_s [1^{\nu-2-2s}], \tag{5.3a}$$

$$\Delta_\pm \Delta_\mp = \sum_s [1^{\nu-1-2s}]. \tag{5.3b}$$

Furthermore, from (4.9)

$$\Delta\square = [\Delta; 1^\nu/Q] = \sum_s [\Delta; 1^{\nu-s}], \tag{5.4a}$$

$$\Delta''\square = [\Delta; 1^\nu/L]'' = \sum_s (-1)^s [\Delta; 1^{\nu-s}]''. \tag{5.4b}$$

Similarly, from (5.2) and (2.4),

$$\Delta\square'' = \Delta^2\Delta'' = \Delta''[1^\nu/S] = \Delta''[1^\nu/MQ],$$

$$\Delta''\square'' = \Delta\Delta''^2 = \Delta[1^\nu/R] = \Delta[1^\nu/LP],$$

so that (4.9) and (2.1) then yield

$$\Delta\square'' = [\Delta; 1^\nu/Q]'' = \sum_s [\Delta; 1^{\nu-s}]'', \tag{5.4c}$$

$$\Delta''\square'' = [\Delta; 1^\nu/L] = \sum_s (-1)^s [\Delta; 1^{\nu-s}]. \tag{5.4d}$$

From these results it follows that

$$\Delta_\pm\square_\pm = \sum_s [\Delta; 1^{\nu-2s}]_\pm, \tag{5.5a}$$

$$\Delta_\pm\square_\mp = \sum_s [\Delta; 1^{\nu-1-2s}]_\mp. \tag{5.5b}$$

Finally, the special case of (4.14) with $[\lambda] = \square$ and the use of (5.2), together with the rules for evaluating Kronecker products of tensor irreps of $O_n$ (Littlewood 1958), lead to

$$\square^2 = \sum_s \left( [2^{\nu-2s}, 1^{2s}] + 2\sum_t [2^{\nu-1-2s-t}, 1^{2s}] \right), \tag{5.6a}$$

$$\square''^2 = \sum_s \left( [2^{\nu-2s}, 1^{2s}] - 2\sum_t (-1)^t [2^{\nu-1-2s-t}, 1^{2s}] \right), \tag{5.6b}$$

$$\square\square'' = \sum_s [2^{\nu-2s}, 1^{2s}]'', \tag{5.6c}$$

and thence to (Wybourne and Butler 1969)

$$\square_{\pm}\square_{\pm} = \sum_s \left([2^{\nu-2s}, 1^{2s}]_{\pm} + \sum_t [2^{\nu-2-2s-2t}, 1^{2s}]\right), \tag{5.7a}$$

$$\square_{\pm}\square_{\mp} = \sum_s \left(\sum_t [2^{\nu-1-2s-2t}, 1^{2s}]\right). \tag{5.7b}$$

## 6. General Kronecker products

With the basic Kronecker products established, it becomes possible to consider arbitrary Kronecker products for both $O_n$ and $SO_n$. Littlewood (1958) has shown that Kronecker products of tensor irreps of $O_n$ may be evaluated through the use of the formula

$$[\lambda][\mu] = \sum_\zeta [(\lambda/\zeta) \cdot (\mu/\zeta)] \tag{6.1}$$

and, where necessary, the equivalences given by (3.2*a*).

Further products may then be evaluated through the use of (3.4), (3.5) and (5.1). It should be noted that for the groups $O_n$, if $n$ is *even*,

$$\Delta^2 = Q = Q^* \tag{6.2a}$$

and if $n$ is *odd*

$$\Delta^2 = \tfrac{1}{2} Q^* \tag{6.2b}$$

since $[1^r] = [1^{n-r}]^* = \{1^r\}$, and $\{1^r\} = 0$ for $r > n$. It is then straightforward to show that for $O_{2\nu}$

$$[\Delta; \lambda][\mu] = \sum_\zeta [\Delta; (\lambda/\zeta) \cdot (\mu/\zeta Q)], \tag{6.3a}$$

$$[\Delta; \lambda][\Delta; \mu] = \sum_\zeta [Q \cdot (\lambda/\zeta) \cdot (\mu/\zeta)], \tag{6.3b}$$

whilst for $O_{2\nu+1}$

$$[\Delta; \lambda][\mu] = \sum_\zeta [\Delta; (\lambda/\zeta) \cdot (\mu/\zeta Q^*)], \tag{6.4a}$$

$$[\Delta; \lambda][\Delta; \mu] = \sum_\zeta \tfrac{1}{2}[Q^* \cdot (\lambda/\zeta) \cdot (\mu/\zeta)]. \tag{6.4b}$$

Whilst (6.3*a*) and (6.4*a*) cannot be improved upon, the same is not true of (6.3*b*) and (6.4*b*), since these expressions involve sums over the infinite number of terms in the series $Q$ and $Q^*$. In the case of (6.3*a*) and (6.4*a*) this is no problem, and the non-zero contributions are limited by the number of parts of the partition $\mu$. It is the modification rules (3.2*b*) which ultimately place a limit on the number of non-zero terms in (6.3*b*) and (6.4*b*). It may be shown firstly that the modification rules appropriate to any term $\rho$ appearing in the products $(\lambda/\zeta) \cdot (\mu/\zeta)$ of (6.3*b*) and 6.4*b*) are

$$(\rho) = (-1)^{x-1}(\rho - h)^* \qquad \text{with } h = 2q - n - 1 = 2q - 2\nu - 1 \tag{6.5a}$$

and

$$(\rho)=(-1)^{x-1}(\rho-h) \qquad \text{with } h=2q-n-1=2q-2\nu-2 \tag{6.5b}$$

respectively, where $q$ is the number of parts of the partition $\rho$. Secondly, making use of these rules to restrict consideration to cases for which $q \leqslant \nu$, for $O_{2\nu}$

$$[Q\cdot\rho]=[1^{\nu};\rho/Q]+\sum_{s=0}^{\nu-1-q}([1^{\nu-1-s};\rho/Q]+[1^{\nu-1-s};\rho/Q]^{*}) \tag{6.6a}$$

and for $O_{2\nu+1}$

$$\tfrac{1}{2}[Q^{*}\cdot\rho]=\sum_{s=0}^{\nu-q}[1^{\nu-s};\rho/Q^{*}]^{(*)^{\nu-s}}. \tag{6.6b}$$

It then follows that (6.3$b$) and (6.4$b$) may be replaced for $O_{2\nu}$ by

$$[\Delta;\lambda][\Delta;\mu]=\sum_{\zeta}\Big\{[1^{\nu};((\lambda/\zeta)\cdot(\mu/\zeta))_q/Q] + \sum_{s=0}^{\nu-1-q}([1^{\nu-1-s};((\lambda/\zeta)\cdot(\mu/\zeta))_q/Q]+[1^{\nu-1-s};((\lambda/\zeta)\cdot(\mu/\zeta))_q/Q]^{*})\Big\} \tag{6.7a}$$

and for $O_{2\nu+1}$ by

$$[\Delta;\lambda][\Delta;\mu]=\sum_{\zeta}\Big\{\sum_{s=0}^{\nu-q}[1^{\nu-s};((\lambda/\zeta)\cdot(\mu/\zeta))_q/Q^{*}]^{(*)^{\nu-s}}\Big\}, \tag{6.7b}$$

where the subscript $q$ has been included in the factor $((\lambda/\zeta)\cdot(\mu/\zeta))_q$ as a reminder that each term $\rho$ in this factor must be modified, using (6.5$a$) and (6.5$b$), to give a term specified by a partition into $q$ parts with $q \leqslant \nu$. This has to be carried out prior to the division by $Q$ or by $Q^{*}$. No further modification rules are then necessary.

In the case of $SO_{2\nu+1}$ it is only necessary to delete * from (6.4$a$) and (6.7$b$). For $SO_{2\nu}$, however, Kronecker products involving difference characters are also required. From (4.8) and (4.9)

$$[\Delta;\lambda][\mu]=\sum_{\zeta}[\Delta;(\lambda/\zeta)\cdot(\mu/\zeta Q)], \tag{6.8a}$$

$$[\Delta;\lambda]''[\mu]=\sum_{\zeta}[\Delta;(\lambda/\zeta)\cdot(\mu/\zeta L)]'', \tag{6.8b}$$

so that from (4.6) (King 1975a)

$$[\Delta;\lambda]_{\pm}[\mu]=\sum_{\zeta,s}[\Delta;(\lambda/\zeta)\cdot(\mu/\zeta 1^{s})]_{\pm(-)^{s}}, \tag{6.8c}$$

where it may be necessary to use the modification rule (4.16$d$). Continuing the analogues of (6.3$b$) and (6.4$b$) for $SO_{2\nu}$ are

$$[\Delta;\lambda][\Delta;\mu]=\sum_{\zeta}[Q\cdot(\lambda/\zeta)\cdot(\mu/\zeta)], \tag{6.9a}$$

$$[\Delta;\lambda]''[\Delta;\mu]''=\sum_{\zeta}(-1)^{\nu}[L\cdot(\lambda/\zeta)\cdot(\mu/\zeta)], \tag{6.9b}$$

whilst

$$[\Delta;\lambda]''[\Delta;\mu]=\sum_{\zeta}[1^{\nu};(\lambda/\zeta Q)\cdot(\mu/\zeta L)]'' \tag{6.9c}$$

subject to the modification rule (4.16*f*). Once again the factors of $(\lambda/\zeta)\cdot(\mu/\zeta)$ appearing in (6.9*a*) and (6.9*b*) should be modified in accordance, this time, with the rules

$$\rho=(-1)^{x}(\rho-h) \qquad \text{with } h=2q-n-1=2q-2\nu-1 \tag{6.10a}$$

and

$$\rho=(-1)^{x-1}(\rho-h) \qquad \text{with } h=2q-n-1=2q-2\nu-1 \tag{6.10b}$$

respectively. Having carried this out to enable further consideration to be restricted to those cases for which $q\leqslant\nu$, the appropriate $SO_{2\nu}$ identities are

$$[Q\cdot\rho]=[1^{\nu};\rho/Q]+2\sum_{s=0}^{\nu-1-q}[1^{\nu-1-s};\rho/Q], \tag{6.11a}$$

$$(-1)^{\nu}[L\cdot\rho]=[1^{\nu};\rho/L]-2\sum_{s=0}^{\nu-1-q}(-1)^{s}[1^{\nu-1-s};\rho/L]. \tag{6.11b}$$

These yield

$$[\Delta;\lambda][\Delta;\mu]=\sum_{\zeta}\left\{[1^{\nu};((\lambda/\zeta)\cdot(\mu/\zeta))_{q}/Q]+2\sum_{s=0}^{\nu-1-q}[1^{\nu-1-s};((\lambda/\zeta)\cdot(\mu/\zeta))_{q}/Q]\right\}, \tag{6.12a}$$

$$[\Delta;\lambda]''[\Delta;\mu]''=\sum_{\zeta}\left\{[1^{\nu};((\lambda/\zeta)\cdot(\mu/\zeta))_{q}/L]\right.$$
$$\left.-2\sum_{s=0}^{\nu-1-q}(-1)^{s}[1^{\nu-1-s};((\lambda/\zeta)\cdot(\mu/\zeta))_{q}/L]\right\}, \tag{6.12b}$$

where once again the subscript $q$ serves as a reminder that (6.10*a*) and (6.10*b*) should be used, if necessary, before dividing by $Q$ and $L$ respectively.

From (4.7*b*)

$$[\Delta;\lambda]_{\pm}[\Delta;\mu]_{\pm}=\tfrac{1}{4}\left\{[\Delta;\lambda][\Delta;\mu]+[\Delta;\lambda]''[\Delta;\mu]''\pm[\Delta;\lambda]''[\Delta;\mu]\pm[\Delta;\lambda][\Delta;\mu]''\right\}, \tag{6.13a}$$

$$[\Delta;\lambda]_{\pm}[\Delta;\mu]_{\mp}=\tfrac{1}{4}\left\{[\Delta;\lambda][\Delta;\mu]-[\Delta;\lambda]''[\Delta;\mu]''\pm[\Delta;\lambda]''[\Delta;\mu]\mp[\Delta;\lambda][\Delta;\mu]''\right\}, \tag{6.13b}$$

so that using (6.9*c*) and (6.12), these products may be evaluated.

The only thing which prevents a general formula being written for these products is the distinction between the modification rules to be used in (6.9*c*), (6.12*a*) and (6.12*b*), i.e. the distinction between applying (4.10*e*) after division by $Q$ and $L$ in (6.9*c*), and applying (6.10*a*) and (6.10*b*) before division by $Q$ and $L$ in (6.11*a*) and (6.12*b*). It is a remarkable fact, however, that these modification rules lead to identical sets of terms $[\square;\sigma]$ and $[\square;\sigma]''$ in (6.9*c*), (6.12*a*) and (6.12*b*), differing only in sign factors $\pm 1$ multiplying each of them.

This is the end of the success story, in that obtaining similar concise general formulae for the remaining products of $SO_{2\nu}$ appears to be a formidable problem.

For example, from (successively) (4.13*b*), (6.1), (4.14*b*), (2.5) with $Z=V$ and (2.3), we find

$$[\square;\lambda]''[\mu]=\sum_{\zeta}[\square;(\lambda/\zeta)\cdot(\mu/\zeta V)]''. \tag{6.14}$$

Unfortunately the double summations in (4.13*a*) and (4.14*a*) inhibit the derivation of any similar formulae for $[\square;\lambda][\mu]$.

Nevertheless $[\square;\lambda]\,[\mu]$ can of course be evaluated directly from (6.1) with $\lambda$ replaced by $\square;\lambda$. The result in combination with (6.14) then yields $[\square;\lambda]_{\pm}\,[\mu]$ as required.

Similarly $[\square;\lambda][\square;\mu]$ may be evaluated from (6.1) and $[\square;\lambda]''[\square;\mu]$ from (6.14), but (4.13*b*), together with (6.1), gives, after using (2.5) with $Z=W$,

$$[\square;\lambda]''[\square;\mu]''=\square''^{2}\sum_{\zeta}[((\lambda/\zeta)\cdot(\mu/\zeta))/W],\tag{6.15}$$

where the complication is the inclusion of the $\nu$-dependent factor prefixing the $\nu$-independent summation over $\zeta$. This factor is given by (5.6*b*) and then, via (6.1) again, it is possible to evaluate the product (6.15) explicitly, and thus to obtain by means of the formulae (6.13) with $\Delta$ replaced by $\square$ the products $[\square;\lambda]_{\pm}[\square;\mu]_{\pm}$ and $[\square;\lambda]_{\pm}[\square;\mu]_{\mp}$.

Finally, we can evaluate $[\Delta;\lambda][\square;\mu]$ and $[\Delta;\lambda]''[\square;\mu]$, using (6.8) with $\mu$ replaced by $\square;\mu$, whilst following what is now a familiar procedure leads to the formulae

$$[\Delta;\lambda][\square;\mu]''=\Delta\square''\sum_{\zeta}[((\lambda/\zeta)\cdot(\mu/\zeta M))/P],\tag{6.16a}$$

$$[\Delta;\lambda]''[\square;\mu]''=\Delta''\square''\sum_{\zeta}[((\lambda/\zeta)\cdot(\mu/\zeta P))/M],\tag{6.16b}$$

where the prefixes to the summations over $\zeta$ are given by (5.4). Manipulations with (4.8), (4.9), (6.1), (2.1) and (2.5) with $Z=Q$ and $Z=L$ then yield

$$[\Delta;\lambda][\square;\mu]''=\sum_{s,\zeta}[\Delta;(1^{\nu-s}/Q)\cdot(((\lambda/\zeta L)\cdot(\mu/\zeta))/1^{s}P)],\tag{6.17a}$$

$$[\Delta;\lambda]''[\square;\mu]''=\sum_{s,\zeta}[\Delta;(1^{\nu-s}/L)\cdot(((\lambda/\zeta Q)\cdot(\mu/\zeta))/1^{s}M)]''.\tag{6.17b}$$

From (6.16) and (6.17) it is then possible to evaluate $[\Delta;\lambda]_{\pm}[\square;\mu]_{\pm}$ and $[\Delta;\lambda]_{\pm}[\square;\mu]_{\mp}$. This completes the task of devising expressions which lead to an essentially mechanical evaluation of any Kronecker product of irreps of $O_n$ or $SO_n$ for both $n=2\nu$ and $n=2\nu+1$.

## 7. Resolution of Kronecker powers

Having developed methods of reducing all Kronecker products of $O_n$ and $SO_n$ into irreducible parts, we are faced with the problem of resolving Kronecker powers. That is, given the $m$th power of some representation with character $\lambda$, it is possible to resolve it into its various symmetrised $m$th powers with characters, denoted by $\lambda\otimes\{\mu\}$ in accordance with the reduction

$$\lambda^{m}=\sum_{\mu}f^{\mu}\lambda\otimes\{\mu\}\tag{7.1}$$

where $\mu$ is a partition of $m$ and $f^{\mu}$ is the dimension or degree of the irrep of the symmetric group $S_m$ labelled by $\mu$. The symbol $\otimes$ denotes the operation of plethysm (Littlewood 1950, Wybourne 1970) which was introduced in an extension of the algebra of $S$-functions, but which is intimately associated with group–subgroup reductions (Wybourne and Butler 1969, Butler and King 1973). It gives the branching rule appropriate to the restriction from $U_N$ to a subgroup G, where G possesses a unitary representation with character $\lambda$ of dimension $D(\lambda)=N$, in accordance with the prescription

$$U_N \downarrow G \qquad \{\mu\} \downarrow \lambda \otimes \{\mu\}, \tag{7.2a}$$

where the $S$-function $\{\mu\}$ is the character of an irrep of $U_N$ of dimension $D_N\{\mu\}$. It follows that necessarily

$$D(\lambda \otimes \{\mu\}) = D_N\{\mu\}. \tag{7.2b}$$

Littlewood (1950, p 290) has developed the algebra of plethysms which is governed by the rules

$$A \otimes (B \pm C) = A \otimes B \pm A \otimes C, \tag{7.3a}$$

$$A \otimes (BC) = (A \otimes B)(A \otimes C), \tag{7.3b}$$

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C, \tag{7.3c}$$

$$(A+B) \otimes \{\mu\} = \sum_{\zeta} (A \otimes \{\mu/\zeta\})(B \otimes \{\zeta\}), \tag{7.3d}$$

$$(A-B) \otimes \{\mu\} = \sum_{\zeta} (-1)^z (A \otimes \{\mu/\zeta\})(B \otimes \{\tilde{\zeta}\}), \tag{7.3e}$$

$$(AB) \otimes \{\mu\} = \sum_{\rho} (A \otimes \{\rho\})(B \otimes \{\mu \circ \rho\}). \tag{7.3f}$$

Here $\circ$ signifies an inner product of $S$-functions so that $\mu \circ \rho$ is the Kronecker product of irreps of $S_m$ labelled $\mu$ and $\rho$ which are both partitions of $m$.

Not all plethysms $A \otimes \{\mu\}$ for different $\mu$ are independent of one another. For example, from (7.3$a$) and (7.3$b$)

$$A \otimes \{1^2\} = A^2 - A \otimes \{2\}, \tag{7.4}$$

whilst

$$A \otimes \{3\} = (A \otimes \{2\})A - A \otimes \{21\}, \tag{7.5a}$$

$$A \otimes \{1^3\} = (A \otimes \{1^2\})A - A \otimes \{21\}. \tag{7.5b}$$

More significantly, from (7.3$c$)

$$A \otimes (\{2\} \otimes \{2\}) = (A \otimes \{2\}) \otimes \{2\} = A \otimes (\{4\} + \{2^2\}), \tag{7.6a}$$

$$A \otimes (\{2\} \otimes \{1^2\}) = (A \otimes \{2\}) \otimes \{1^2\} = A \otimes \{31\}, \tag{7.6b}$$

$$A \otimes (\{1^2\} \otimes \{2\}) = (A \otimes \{1^2\}) \otimes \{2\} = A \otimes (\{2^2\} + \{1^4\}), \tag{7.6c}$$

$$A \otimes (\{1^2\} \otimes \{1^2\}) = (A \otimes \{1^2\}) \otimes \{1^2\} = A \otimes \{21^2\}, \tag{7.6d}$$

where use has been made of some very simple plethysms, and from (7.3$b$)

$$A \otimes (\{3\} \cdot \{1\}) = (A \otimes \{3\})A = A \otimes (\{4\} + \{31\}), \tag{7.7a}$$

$$A \otimes (\{1^3\} \cdot \{1\}) = (A \otimes \{1^3\})A = A \otimes (\{21^2\} + \{1^4\}), \tag{7.7b}$$

$$A \otimes (\{2\} \cdot \{2\}) = (A \otimes \{2\})^2 = A \otimes (\{4\} + \{31\} + \{2^2\}), \tag{7.7c}$$

leading to the basic identities

$$A \otimes \{4\} = (A \otimes \{3\})A - (A \otimes \{2\}) \otimes \{1^2\}, \tag{7.8a}$$

$$A \otimes \{31\} = (A \otimes \{2\}) \otimes \{1^2\}, \tag{7.8b}$$

$$A \otimes \{2^2\} = (A \otimes \{2\})^2 - (A \otimes \{3\})A, \tag{7.8c}$$

$$A \otimes \{21^2\} = (A \otimes \{1^2\}) \otimes \{1^2\}, \tag{7.8d}$$

$$A \otimes \{1^4\} = (A \otimes \{1^3\})A - (A \otimes \{1^2\}) \otimes \{1^2\}. \tag{7.8e}$$

These allow all Kronecker powers of degree four to be resolved from a knowledge of lower-degree powers and products. Indeed, to resolve all Kronecker powers of $A$ of degrees two, three and four it is only necessary to evaluate $A \otimes \{2\}$ and $A \otimes \{21\}$, for example.

The key to the resolution of the Kronecker powers of irreps of $O_n$ is the recognition that the relationships between characters of $O_n$ and $U_n$ (Littlewood 1950, p 240) is

$$U_n \downarrow O_n \qquad \{\lambda\} \downarrow [\lambda/D], \tag{7.9a}$$

$$U_n \uparrow O_n \qquad [\lambda] \uparrow \{\lambda/C\}, \tag{7.9b}$$

so that the Kronecker $m$th powers of tensor irreps of $O_n$ are determined by the formulae

$$[\lambda] \otimes \{\mu\} = [(\{\lambda/C\} \otimes \{\mu\})/D] \tag{7.10a}$$

and

$$[\lambda]^* \otimes \{\mu\} = [(\{\lambda/C\} \otimes \{\mu\})/D]^{(*)^m} \tag{7.10b}$$

where $\mu$ is a partition of $m$. This last result follows from the facts that $[\lambda]^* = [0]^*[\lambda]$, $[0]^* = [1^n] = \{1^n\}$ and $\{1^n\} \otimes \{\rho\} = 0$ for $(\rho) \neq (m)$ and $\{1^n\} \otimes \{m\} = \{1^{nm}\} = \{1^n\}^m = [0]^{(*)^m}$, together with (7.3$f$).

In the case of $SO_{2\nu+1}$ (7.10$a$) suffices. For $SO_{2\nu}$, if $\lambda_\nu = 0$ then (7.10$a$) again suffices, provided (4.6$a$) is applied where appropriate to characters appearing on the right-hand side of (7.10$a$). Of course, in many cases the modification rules (3.2$a$) may also be required.

Kronecker powers of spin irreps of $O_n$ may be resolved by recalling (7.3$f$) and making use of (3.4$a$) and (3.5$a$), which lead for $O_{2\nu}$ to

$$[\Delta; \lambda] \otimes \{\mu\} = \sum_\rho [\Delta \otimes \{\rho\}][(\{\lambda/E\} \otimes \{\mu \circ \rho\})/D] \tag{7.11a}$$

and for $O_{2\nu+1}$ to

$$[\Delta; \lambda] \otimes \{\mu\} = \sum_\rho [\Delta \otimes \{\rho\}][(\{\lambda/E^*\} \otimes \{\mu \circ \rho\})/D] \tag{7.11b}$$

where $E^* = P^*C$ so that the only difference between (7.11$a$) and (7.11$b$) is that the latter indicates a factor $[0]^{(*)^e}$ for each partition $\varepsilon$ of $e$ appearing in the series $E$.

Again, for $SO_{2\nu+1}$ (7.11$b$) suffices with the * removed to give (7.11$a$), but for $SO_{2\nu}$ it is necessary to resort to difference characters. In addition to (7.10$a$), including the case for which $[\lambda]$ is replaced by $[\Box; \lambda]$, and (7.11$a$) we need the results

$$[\Delta; \lambda]'' \otimes \{\mu\} = \sum_\rho [\Delta'' \otimes \{\rho\}][(\{\lambda/G\} \otimes \{\mu \circ \rho\})/D], \tag{7.12a}$$

$$[\Box; \lambda]'' \otimes \{\mu\} = \sum_\rho [\Box'' \otimes \{\rho\}][(\{\lambda/A\} \otimes \{\mu \circ \rho\})/D], \tag{7.12b}$$

where use has been made of (4.8*b*), (4.13*b*), (7.3*f*) and the *S*-function series identities of § 2.

It is worth pointing out that the automorphism $^\dagger$ of $SO_{2\nu}$ is such that by virtue of (4.5), and the definitions (4.6) and (4.11),

$$[\Delta;\lambda]''^\dagger=-[\Delta;\lambda]'' \qquad \text{and} \qquad [\Box;\lambda]''^\dagger=-[\Box;\lambda]''. \tag{7.13}$$

Hence from (7.3*e*)

$$[\Delta;\lambda]''\otimes\{\tilde\mu\}=(-[\Delta;\lambda]''^\dagger)\otimes\{\tilde\mu\}=((-[\Delta;\lambda]'')\otimes\{\tilde\mu\})^\dagger=(-1)^m([\Delta;\lambda]''\otimes\{\mu\})^\dagger \tag{7.14a}$$

and similarly

$$[\Box;\lambda]''\otimes\{\tilde\mu\}=(-1)^m([\Box;\lambda]''\otimes\{\mu\})^\dagger \tag{7.14b}$$

whilst, more obviously,

$$[\Delta;\lambda]_\pm\otimes\{\mu\}=([\Delta;\lambda]^\dagger_\mp)\otimes\{\mu\}=([\Delta;\lambda]_\mp\otimes\{\mu\})^\dagger \tag{7.14c}$$

and

$$[\Box;\lambda]_\pm\otimes\{\mu\}=([\Box;\lambda]_\mp{}^\dagger)\otimes\{\mu\}=([\Box;\lambda]_\mp\otimes\{\mu\})^\dagger. \tag{7.14d}$$

These identities reduce the number of separate evaluations which need to be made or tabulated.

From a knowledge of the plethysms involving $[\Delta;\lambda]$, $[\Delta;\lambda]''$, $[\Box;\lambda]$ and $[\Box;\lambda]''$ the plethysms $[\Delta;\lambda]_\pm\otimes\{\mu\}$ and $[\Box;\lambda]_\pm\otimes\{\mu\}$ may be evaluated by noting (4.7*b*) and (4.11*c*), and using (7.3*d*) and (7.3*c*) to give, denoting $[\Delta;\lambda]_\pm$ and $[\Box;\lambda]_\pm$ simply by $[\lambda]_\pm$, for convenience:

$$([\lambda]_+ + [\lambda]_+)\otimes\{\mu\}=\sum_\pi([\lambda]_+\otimes\{\mu/\pi\})([\lambda]_+\otimes\{\pi\})=\sum_\zeta([\lambda]\otimes\{\mu/\zeta\})([\lambda]''\otimes\{\zeta\}), \tag{7.15a}$$

$$\begin{aligned}([\lambda]_- + [\lambda]_-)\otimes\{\mu\}&=\sum_\pi([\lambda]_-\otimes\{\mu/\pi\})([\lambda]_-\otimes\{\pi\})\\ &=\sum_\zeta(-1)^z([\lambda]\otimes\{\mu/\zeta\})([\lambda]''\otimes\{\tilde\zeta\}),\end{aligned} \tag{7.15b}$$

where the first forms of (7.15*a*) and (7.15*b*) include $2([\lambda]_+\otimes\{\mu\})$ and $2([\lambda]_-\otimes\{\mu\})$ respectively, along with plethysms appearing in lower Kronecker powers than the *m*th, where $\mu$ is a partition of $m$. Thus, for example, we find for $m=3$

$$\begin{aligned}[\lambda]_+\otimes\{21\}&=\tfrac12(([\lambda]+[\lambda]'')\otimes\{21\})-[\lambda]_+^3\\ &=\tfrac12([\lambda]\otimes\{21\}+[\lambda]''\otimes\{21\})-\tfrac18([\lambda]^3-[\lambda]^2[\lambda]''+[\lambda][\lambda]''^2-[\lambda]''^3).\end{aligned} \tag{7.16}$$

It follows that we have a systematic procedure for resolving all Kronecker powers of irreps of $O_n$ and $SO_n$, provided that the Kronecker powers of the basic irreps with characters $\Delta$, $\Delta''$ and $\Box''$ may be resolved. In the case of $\Box''$ there is no problem, in principle, in that from (5.2*c*) and (7.3*f*)

$$\Box''\otimes\{\mu\}=(\Delta\Delta'')\otimes\{\mu\}=\sum_\rho(\Delta\otimes\{\rho\})(\Delta''\otimes\{\mu\circ\rho\}). \tag{7.17}$$

Incidentally, from (7.10*a*),

$$\Box\otimes\{\mu\}=[1^\nu]\otimes\{\mu\}=[(\{1^\nu\}\otimes\{\mu\})/D]. \tag{7.18}$$

The final task is then that of evaluating the basic spin plethysms $\Delta\otimes\{\mu\}$ and $\Delta''\otimes\{\mu\}$.

## 8. Resolution of the basic Kronecker squares

Littlewood (1947) has given the complete resolution of the Kronecker square of the basic spin irrep of $O_n$ into its symmetric ($\Delta \otimes \{2\}$) and antisymmetric ($\Delta \otimes \{1^2\}$) parts. For $O_{2\nu}$ he found

$$\Delta \otimes \{2\} = [1^\nu] + \sum_x ([1^{\nu-1-4x}] + [1^{\nu-3-4x}]^* + [1^{\nu-4-4x}] + [1^{\nu-4-4x}]^*), \tag{8.1a}$$

$$\Delta \otimes \{1^2\} = \sum_x ([1^{\nu-1-4x}]^* + [1^{\nu-2-4x}] + [1^{\nu-2-4x}]^* + [1^{\nu-3-4x}]), \tag{8.1b}$$

whilst for $O_{2\nu+1}$ the cases for $\nu$ *even* and $\nu$ *odd* must be distinguished, giving for $\nu$ *even*

$$\Delta \otimes \{2\} = [1^\nu] + \sum_x ([1^{\nu-3-4x}]^* + [1^{\nu-4-4x}]), \tag{8.2a}$$

$$\Delta \otimes \{1^2\} = \sum_x ([1^{\nu-1-4x}]^* + [1^{\nu-2-4x}]), \tag{8.2b}$$

and for $\nu$ *odd*

$$\Delta \otimes \{2\} = [1^\nu]^* + \sum_x ([1^{\nu-3-4x}] + [1^{\nu-4-4x}]^*), \tag{8.3a}$$

$$\Delta \otimes \{1^2\} = \sum_x ([1^{\nu-1-4x}] + [1^{\nu-2-4x}]^*). \tag{8.3b}$$

In the case of $SO_{2\nu+1}$ it is only necessary to set $[\lambda]^* = [\lambda]$ in (8.2) and (8.3), which then yield identical formulae for $\nu$ even and odd.

For $SO_{2\nu}$ the same procedure yields

$$\Delta \otimes \{2\} = [1^\nu] + \sum_x ([1^{\nu-1-4x}] + [1^{\nu-3-4x}] + 2[1^{\nu-4-4x}]), \tag{8.4a}$$

$$\Delta \otimes \{1^2\} = \sum_x ([1^{\nu-1-4x}] + 2[1^{\nu-2-4x}] + [1^{\nu-3-4x}]). \tag{8.4b}$$

In order to cope with the difference characters, it is helpful to note first that

$$\Delta \otimes \{1^2\} = (\Delta_+ + \Delta_-) \otimes \{1^2\} = \Delta_+ \otimes \{1^2\} + \Delta_+\Delta_- + \Delta_- \otimes \{1^2\},$$

so that from (7.4*b*) and (5.3*b*)

$$\Delta_+ \otimes \{1^2\} + \Delta_- \otimes \{1^2\} = 2 \sum_x [1^{\nu-2-4x}].$$

Each term of the right-hand side is invariant under the automorphism $^\dagger$. This implies that the same is true for both terms on the left-hand side. Hence

$$\Delta_+ \otimes \{1^2\} = (\Delta_+ \otimes \{1^2\})^\dagger = \Delta_- \otimes \{1^2\}.$$

Therefore

$$\begin{aligned}\Delta'' \otimes \{2\} &= (\Delta_+ - \Delta_-) \otimes \{2\} = \Delta_+ \otimes \{2\} - \Delta_+\Delta_- + \Delta_- \otimes \{1^2\} \\ &= \Delta_+^2 - \Delta_+\Delta_- = \tfrac{1}{2}(\Delta''^2 + \Box'')\end{aligned}$$

and likewise

$$\Delta'' \otimes \{1^2\} = \tfrac{1}{2}(\Delta''^2 - \Box'')$$

with the terms in $\Delta''^2$ following from (5.2$b$). Hence (Wybourne and Butler 1969)

$$\Delta'' \otimes \{2\} = [1^\nu]_+ - \sum_x (-1)^x [1^{\nu-1-x}], \tag{8.5a}$$

$$\Delta'' \otimes \{1^2\} = [1^\nu]_- - \sum_x (-1)^x [1^{\nu-1-x}], \tag{8.5b}$$

and combining this result with (8.4) yields

$$\Delta_\pm \otimes \{2\} = [1^\nu]_\pm + \sum_x [1^{\nu-4-4x}], \tag{8.6a}$$

$$\Delta_\pm \otimes \{1^2\} = \sum_x [1^{\nu-2-4x}], \tag{8.6b}$$

in conformity with the results of Littlewood (1947).

In the case of the characters $\Box$ and $\Box''$, the resolution of $\Box^2$ follows from a knowledge of the plethysm (Littlewood 1943)

$$\Box \otimes \{2\} = \{1^\nu\} \otimes \{2\} = \sum_s \{2^{\nu-s}, 1^{2s}\} = \sum_{s,t} [2^{\nu-s-t}, 1^{2s}]$$

and the use of the modification rule (3.2$a$), whilst the resolution of $\Box''^2$ proceeds by noting (5.2$c$) and writing

$$\Box'' \otimes \{2\} = (\Delta\Delta'') \otimes \{2\} = (\Delta \otimes \{2\})(\Delta'' \otimes \{2\}) + (\Delta \otimes \{1^2\})(\Delta'' \otimes \{1^2\}). \tag{8.7}$$

It then may be seen that

$$\Box \otimes \{2\} = \sum_s \Big\{[2^{\nu-4s}, 1^{4s}] + \sum_t ([2^{\nu-1-4s-2t}, 1^{4s}] + 2[2^{\nu-2-4s-2t}, 1^{4s}] + [2^{\nu-3-4s-2t}, 1^{2+4s}])\Big\}, \tag{8.8a}$$

$$\Box \otimes \{1^2\} = \sum_s \Big\{[2^{\nu-2-4s}, 1^{2+4s}] + \sum_t ([2^{\nu-1-4s-2t}, 1^{4s}] + [2^{\nu-3-4s-2t}, 1^{2+4s}] + 2[2^{\nu-4-4s-2t}, 1^{2+4s}])\Big\}, \tag{8.8b}$$

$$\Box'' \otimes \{2\} = \sum_s \Big\{[2^{\nu-2s}, 1^{2s}]_{+(-)^s} - \sum_t (-1)^t [2^{\nu-1-2s-t}, 1^{2s}]\Big\}, \tag{8.9a}$$

$$\Box'' \otimes \{1^2\} = \sum_s \Big\{[2^{\nu-2s}, 1^{2s}]_{-(-)^s} - \sum_t (-1)^t [2^{\nu-1-2s-t}, 1^{2s}]\Big\}. \tag{8.9b}$$

Hence

$$\Box_\pm \otimes \{2\} = \sum_s \Big\{[2^{\nu-4s}, 1^{4s}]_\pm + \sum_t [2^{\nu-2-4s-2t}, 1^{4s}]\Big\}, \tag{8.10a}$$

$$\Box_\pm \otimes \{1^2\} = \sum_s \Big\{[2^{\nu-2-4s}, 1^{2+4s}]_\pm + \sum_t [2^{\nu-4-4s-2t}, 1^{2+4s}]\Big\}. \tag{8.10b}$$

## 9. Resolution of the basic spin Kronecker cubes

The evaluation of the plethysms $\Delta \otimes \{\mu\}$ for the groups $O_{2\nu+1}$ or $O_{2\nu}$ is equivalent, by virtue of (7.2), to determining the reduction of the irrep of $U_{2^\nu}$ having character $\{\mu\}$ into irreps of $O_{2\nu+1}$ or $O_{2\nu}$ respectively. The dimensions of the irreps are readily found. In particular,

$$D_{2^\nu}\{3\} = 2^\nu(2^\nu+1)(2^\nu+2)/6, \tag{9.1a}$$

$$D_{2^\nu}\{21\} = 2^\nu(2^\nu+1)(2^\nu-1)/3, \tag{9.1b}$$

$$D_{2^\nu}\{1^3\} = 2^\nu(2^\nu-1)(2^\nu-2)/6 = \binom{2^\nu}{3} \tag{9.1c}$$

whilst

$$D_{2\nu+1}[1^x] = \binom{2\nu+1}{x}, \tag{9.2a}$$

$$D_{2\nu}[1^x] = \binom{2\nu}{x}, \tag{9.2b}$$

and, of course,

$$D_{2\nu+1}[\Delta] = D_{2\nu}[\Delta] = 2^\nu, \tag{9.3a}$$

$$D_{2\nu}[\Delta]'' = D_{2\nu}[\square]'' = 0, \tag{9.3b}$$

$$D_{2\nu}[\Delta]_\pm = 2^{\nu-1}.$$

In evaluating the resolved Kronecker cube of $\Delta$, we endeavour to express the results as a product of the basic spin character $\Delta$ and a series of characters of $O_n$ of the generic type $[1^x]$. It is a non-trivial task to distinguish between mutually associate pairs of irreps, so that from now on we limit attention to the groups $SO_n$. In the case of $n=2\nu+1$ and $n=2\nu$, with $\nu=1,2,\ldots$, the results for $\Delta\otimes\{21\}$ may be readily evaluated, exploiting known isomorphisms (automorphisms in the case $n=8$) and explicit evaluation using Kronecker products and dimension checks (computer generated tables in the case $n=10$ by courtesy of Dr P H Butler) together with the branching rules

$$\begin{array}{lll}
U_{2^\nu} & \downarrow SO_{2\nu+1} & \downarrow SO_{2\nu} \\
 & [1] & \downarrow [1]+[0] \\
\{1\} & \downarrow \Delta & \downarrow \Delta = \Delta_+ + \Delta_- \\
 & [0] & \downarrow [0] \\
 & [1^x] & \downarrow [1^x]+[1^{x-1}] \\
\{21\} & \downarrow \Delta\otimes\{21\} & \downarrow \Delta\otimes\{21\}.
\end{array} \tag{9.4}$$

These results for $SO_{2\nu+1}$ with $\nu=1,2,\ldots,5$ are all of the form

$$\Delta\otimes\{21\} = \Delta([1^{\nu-1}]+[1^{\nu-4}]+\ldots)$$

which suggests that in general

$$\Delta\otimes\{21\} = \Delta\sum_x [1^{\nu-1-3x}]. \tag{9.5}$$

This may be checked dimensionally, and is in agreement with the combinatorial identity

$$(2^\nu+1)(2^\nu-1)/3 = \sum_x \binom{2\nu+1}{\nu-1-3x}. \tag{9.6}$$

Analogous identities can be established for the terms in $\Delta\otimes\{3\}$ and $\Delta\otimes\{1^3\}$. Alternatively, and more simply, the use of (7.5) gives the complete results for $SO_{2\nu+1}$:

$$\Delta\otimes\{3\}=\Delta\sum_x([1^{\nu-12x}]-[1^{\nu-1-12x}]+[1^{\nu-3-12x}]+[1^{\nu-8-12x}]-[1^{\nu-10-12x}]+[1^{\nu-11-12x}]), \tag{9.7a}$$

$$\Delta\otimes\{1^3\}=\Delta\sum_x([1^{\nu-2-12x}]-[1^{\nu-4-12x}]+[1^{\nu-5-12x}] +[1^{\nu-6-12x}]-[1^{\nu-7-12x}]+[1^{\nu-9-12x}]). \tag{9.7b}$$

The corresponding results for $SO_{2\nu}$ follow from the branching rules (9.4), which give

$$\Delta\otimes\{3\}=\Delta\Big\{[1^\nu]+\sum_x(-[1^{\nu-2-12x}]+[1^{\nu-3-12x}]+[1^{\nu-4-12x}] +[1^{\nu-8-12x}]+[1^{\nu-9-12x}]-[1^{\nu-10-12x}]+2[1^{\nu-12-12x}])\Big\}, \tag{9.8a}$$

$$\Delta\otimes\{21\}=\Delta\sum_x([1^{\nu-1-3x}]+[1^{\nu-2-3x}]), \tag{9.8b}$$

$$\Delta\otimes\{1^3\}=\Delta\sum_x([1^{\nu-2-12x}]+[1^{\nu-3-12x}]-[1^{\nu-4-12x}] +2[1^{\nu-6-12x}]-[1^{\nu-8-12x}]+[1^{\nu-9-12x}]+[1^{\nu-10-12x}]). \tag{9.8c}$$

It only remains to prove the one result upon which (9.7) and (9.8) depend, namely (9.5). This may be done by induction with respect to $\nu$ and the use of the branchings (King 1975b)

$$U_{2^\nu}\downarrow SO_{2\nu+1}\ \downarrow SO_{2\nu-1} \tag{9.9}$$

$$\{1\}\downarrow\Delta\qquad\downarrow 2\Delta' \tag{9.10a}$$

$$[1]\qquad\downarrow[1]'+2[0]' \tag{9.10b}$$

$$[\lambda]\qquad\downarrow[\lambda/MM]' \tag{9.10c}$$

$$\{21\}\downarrow\Delta\otimes\{21\}\downarrow(2\Delta')\otimes\{21\}, \tag{9.10d}$$

where a prime is used to distinguish characters of $SO_{2\nu-1}$ from those of $SO_{2\nu+1}$. It follows from (9.10$d$) and (7.3$d$) that

$$\Delta\otimes\{21\}\downarrow(\Delta'+\Delta')\otimes\{21\}=2(\Delta'\otimes\{21\}+\Delta'^3). \tag{9.11}$$

Writing

$$\Delta\otimes\{21\}=\Delta X_\nu \qquad\text{and}\qquad \Delta'\otimes\{21\}=\Delta' X'_{\nu-1}, \tag{9.12}$$

this yields, after recourse to (9.10$c$) and the cancellation of a common factor of $2\Delta'$,

$$X_\nu/MM=X'_{\nu-1}+\Delta'^2 \tag{9.13a}$$

and hence the recurrence relation

$$X_\nu=[(X'_{\nu-1}+\Delta'^2)/LL] \tag{9.13b}$$

where of course $[\ldots]'/LL=[\ldots/LL]$. Assuming as the basis of an induction argument that

$$X'_{\nu-1}=\sum_x[1^{\nu-2-3x}]'$$

and taking

$$\Delta'^2 = \sum_x [1^{\nu-1-3x}]'$$

from (5.1*b*), it is straightforward to show from (9.13*b*) that

$$X_\nu = \sum_x [1^{\nu-1-3x}]$$

as required. The induction argument is then completed by means of the known validity of (9.5) in the cases $\nu = 1$ and 2, thus proving the general validity of (9.5).

Having proved in this way the validity of not only (9.5) but also (9.7) and (9.8), it is then necessary to consider the Kronecker cubes of $\Delta''$. Once again explicit results were obtained for $SO_{2\nu}$ with $\nu = 2, 3, 4$ and 5, giving

$$\begin{array}{ll} SO_4 & -\Delta''([1]-[0]), \\ SO_6 & -\Delta''([1^2]-[1]), \\ SO_8 & -\Delta''([1^3]-[1^2]-[0]), \\ SO_{10} & -\Delta''([1^4]-[1^3]-[1]+[0]), \end{array} \tag{9.14}$$

where, remarkably, the bracketed terms are of total dimension $3^{\nu-1}$ in each case. The general series is then identified as

$$\Delta''\otimes\{21\} = \Delta''\sum_x(-[1^{\nu-1-6x}]+[1^{\nu-2-6x}]+[1^{\nu-4-6x}]-[1^{\nu-5-6x}]) \tag{9.15}$$

by virtue of the combinatorial identity

$$3^{\nu-1} = \sum_x\left\{\binom{2\nu}{\nu-1-6x}-\binom{2\nu}{\nu-2-6x}-\binom{2\nu}{\nu-4-6x}+\binom{2\nu}{\nu-5-6x}\right\}. \tag{9.16}$$

The identities (7.11) then yield, from (8.5),

$$\Delta''\otimes\{3\} = \Delta''\left\{[1^\nu]_+ - \sum_x(-1)^x[1^{\nu-3-3x}]\right\}, \tag{9.17a}$$

$$\Delta''\otimes\{1^3\} = \Delta''\left\{[1^\nu]_- - \sum_x(-1)^x[1^{\nu-3-3x}]\right\}. \tag{9.17b}$$

Finally from (7.9*a*) with $\lambda$ replaced by $\Delta$, and (7.5) yet again, we find

$$\begin{aligned}\Delta_\pm\otimes\{3\} = \Delta_\pm[1^\nu]_\pm + \sum_x\Big\{ &-\Delta_\pm[1^{\nu-2-12x}]+\Delta_\mp[1^{\nu-3-12x}] \\ &+\Delta_\mp[1^{\nu-9-12x}]-\Delta_\pm[1^{\nu-10-12x}]+\Delta_\pm[1^{\nu-12-12x}]\Big\},\end{aligned} \tag{9.18a}$$

$$\Delta_\pm\otimes\{21\} = \sum_x\left\{\Delta_\pm[1^{\nu-2-6x}]-\Delta_\mp[1^{\nu-3-6x}]+\Delta_\pm[1^{\nu-4-6x}]\right\}, \tag{9.18b}$$

$$\begin{aligned}\Delta_\pm\otimes\{1^3\} = \sum_x\Big\{ &\Delta_\mp[1^{\nu-3-12x}]-\Delta_\pm[1^{\nu-4-12x}]+\Delta_\pm[1^{\nu-6-12x}] \\ &-\Delta_\pm[1^{\nu-8-12x}]+\Delta_\mp[1^{\nu-9-12x}]\Big\},\end{aligned} \tag{9.18c}$$

where it is to be noted that $[1^{\nu-p-12x}]$ is always associated with a factor $\Delta_{\pm(-)^p}$. This provides the clue to proving the validity of these results which all depend on the conjecture (9.15). They may equivalently be seen to depend upon (9.18*b*), which may be proved by consideration of a group–subgroup chain once again.

The relevant chain and branching rules are those of (9.4) which imply, using (9.12), that

$$(\Delta_+ + \Delta_-) \otimes \{21\} = (\Delta_+ + \Delta_-)[X_\nu/M] \tag{9.19}$$

with $X_\nu$ determined by (9.5). It follows from (7.3*d*) that

$$\begin{aligned}(\Delta_+ + \Delta_-) \otimes \{21\} &= \Delta_+ \otimes \{21\} + \Delta_+^2\Delta_- + \Delta_-\Delta_+^2 + \Delta_- \otimes \{21\} \\ &= \Delta_+ \otimes \{21\} + \Delta_- \otimes \{21\} + (\Delta_+ + \Delta_-)\Delta_+\Delta_-.\end{aligned} \tag{9.20}$$

We now write

$$\Delta_+ \otimes \{21\} = \Delta_+ Y_\nu + \Delta_- Z_\nu \tag{9.21}$$

where $Y_\nu$ and $Z_\nu$ are tensor characters whose irreducible constituents are necessarily labelled by partitions of $y$ and $z$ respectively, with $y \equiv \nu \pmod 2$ and $z \equiv (\nu-1) \pmod 2$ by virtue of the Kronecker product rules for $SO_{2\nu}$ such as (4.10), (5.3) and (6.1), as well as the modification rules (4.16) which all preserve the ranks of partitions of irreps of $SO_{2\nu}$ mod 2. This implies a similar preservation of rank mod 2 in the case under consideration of Kronecker cubes. Furthermore, the automorphism $^\dagger$ is such that

$$\Delta_- \otimes \{21\} = (\Delta_+ \otimes \{21\})^\dagger = (\Delta_+ Y_\nu + \Delta_- Z_\nu)^\dagger = \Delta_- Y_\nu + \Delta_+ Z_\nu \tag{9.22}$$

so that in (9.20)

$$(\Delta_+ + \Delta_-) \otimes \{21\} = (\Delta_+ + \Delta_-)(Y + Z + \Delta_+\Delta_-), \tag{9.23}$$

giving in conjunction with (9.19) and (5.3*b*)

$$\begin{aligned}Y_\nu + Z_\nu &= [X_\nu/M] - \Delta_+\Delta_- \\ &= \sum_x \left\{[1^{\nu-1-3x}] + [1^{\nu-2-3x}] - [1^{\nu-1-2x}]\right\} \\ &= \sum_x \left\{[1^{\nu-2-6x}] - [1^{\nu-3-6x}] + [1^{\nu-4-6x}]\right\},\end{aligned} \tag{9.24}$$

so that

$$Y_\nu = \sum_x \{[1^{\nu-2-6x}] + [1^{\nu-4-6x}]\} \tag{9.25a}$$

and

$$Z_\nu = \sum_x \{-[1^{\nu-3-6x}]\} \tag{9.25b}$$

as required in (9.18*b*).

From the results on the resolution of the squares and cubes of $\Delta$ and $\Delta''$ it is then straightforward to resolve the same powers of $\square''$ by means of (7.17). Fourth powers may be dealt with through the use of (7.8) which, for $\Delta$ and $\Delta''$, involve the further evaluation of $[1^x]\otimes\{2\}$ and $[1^x]\otimes\{1^2\}$. This is accomplished by noting that

$$\{1^x\}\otimes\{2\}=\sum_s\{2^{x-2s},1^{4s}\},\tag{9.26a}$$

$$\{1^x\}\otimes\{1^2\}=\sum_s\{2^{x-1-2s},1^{2+4s}\},\tag{9.26b}$$

so that from (7.9*a*)

$$[1^x]\otimes\{2\}=\sum_{s,t}[2^{x-2s-t},1^{4s}],\tag{9.27a}$$

$$[1^x]\otimes\{1^2\}=\sum_{s,t}[2^{x-1-2s-t},1^{2+4s}].\tag{9.27b}$$

These allow the full resolution of the Kronecker powers $\Delta^4$ and $\Delta''^4$, and lead via (7.17) to the resolution of $\Box''^4$. While it is possible to produce closed formulae of the type found for Kronecker squares and cubes, they tend to be unwieldy and will not be given here.

## 10. Application to $SO_{10}$

In view of the current interest in $SO_{10}$-based grand unified theories, it is worthwhile illustrating some of the preceding results by explicit application to $SO_{10}$. A short list of dimensions of relevant $SO_{10}$ irreps is given in table 1. Note that in going to $O_{10}$ the dimensions of irreps $[\lambda]$ with $\lambda_5\neq 0$ are double those listed for $[\lambda]_\pm$ in the $SO_{10}$ list, and similarly for all irreps $[\Delta;\lambda]$.

**Table 1.** Dimensions of true and spin irreps of $SO_{10}$.

| $[\lambda]$ | $D_{[\lambda]}$ | $D_{[\Delta;\lambda]_\pm}$ |
|---|---|---|
| [0] | 1 | 16 |
| [1] | 10 | 144 |
| $[1^2]$ | 45 | 560 |
| $[1^3]$ | 120 | 1 200 |
| $[1^4]$ | 210 | 1 440 |
| $[1^5]_\pm$ | 126 | 672 |
| [2] | 54 | 720 |
| [21] | 320 | 3 696 |
| $[21^2]$ | 945 | 8 800 |
| $[21^3]$ | 1 728 | 11 088 |
| $[21^4]_\pm$ | 1 050 | 5 280 |
| $[2^2]$ | 770 | 8 064 |
| $[2^21]$ | 2 970 | 25 200 |
| $[2^21^2]$ | 5 940 | 34 992 |
| $[2^21^3]_\pm$ | 3 696 | 17 280 |
| $[2^3]$ | 4 125 | 30 800 |
| $[2^31]$ | 10 560 | 55 440 |
| $[2^31^2]_\pm$ | 6 930 | 29 568 |
| $[2^4]$ | 8 910 | 39 600 |
| $[2^4 1]_\pm$ | 6 930 | 26 400 |
| $[2^5]_\pm$ | 2 772 | 9 504 |

The basic spin irrep $\Delta$ of $O_{10}$ is of degree $2^5$ and may be embedded in the defining, vector irrep $\{1\}$ of $U_{32}$. The evaluation of the plethysms $\Delta\otimes\{\mu\}$ is equivalent to determining the $U_{32}\downarrow O_{10}$ branching rules. These reductions are given in table 2 for all partitions $\mu$ of rank four or less. The results may be checked by comparison of the dimensions $D_{32}\{\mu\}$ of $U_{32}$ with the sum of the dimensions of the $O_{10}$ irreps contained in the reduction.

The plethysms given in table 2 follow from (8.1) for power two, (9.8) for power three and thence the power four plethysms are evaluated using (7.8) and (10.2).

The independent plethysms $\Delta''\otimes\{\mu\}$ for partitions $\mu$ of rank four or less are listed in table 3. The list may be completed using (7.14). In the case of even-rank partitions $\mu$, we check that the dimension of the plethysm is zero in accordance with (9.3*b*), whilst for

**Table 2.** $U_{32}\downarrow O_{10}$ reductions corresponding to the plethysms $\Delta\otimes\{\mu\}$.

| $D_{\{\mu\}}$ | $U_{32}$ | $O_{10}$ |
|---|---|---|
| 32 | $\{1\}$ | $[\Delta;0]$ |
| 528 | $\{2\}$ | $[0]+2[1]+[1^2]+[1^4]+[1^5]$ |
| 496 | $\{1^2\}$ | $[0]+[1^2]+2[1^3]+[1^4]$ |
| 5 984 | $\{3\}$ | $2[\Delta;0]+2[\Delta;1]+[\Delta;1^2]+[\Delta;1^4]+[\Delta;1^5]$ |
| 10 912 | $\{21\}$ | $4[\Delta;0]+3[\Delta;1]+2[\Delta;1^2]+2[\Delta;1^3]+[\Delta;1^4]$ |
| 4 960 | $\{1^3\}$ | $[\Delta;0]+[\Delta;1]+2[\Delta;1^2]+[\Delta;1^3]$ |
| 52 360 | $\{4\}$ | $2[0]+2[1]+2[1^2]+2[1^3]+3[1^4]+2[1^5]+3[2]+2[21]+2[21^3]+2[21^4]$ $+[2^2]+[2^21^2]+[2^21^3]+[2^4]+[2^41]+[2^5]$ |
| 139 128 | $\{31\}$ | $2[0]+6[1]+8[1^2]+8[1^3]+9[1^4]+5[1^5]+[2]+4[21]+6[21^2]+6[21^3]$ $+3[21^4]+[2^2]+2[2^21]+3[2^21^2]+2[2^21^3]+2[2^31]+2[2^31^2]+[2^4]+[2^41]$ |
| 87 296 | $\{2^2\}$ | $5[0]+4[1]+3[1^2]+6[1^3]+7[1^4]+3[1^5]+4[2]+2[21]+[21^2]+4[21^3]$ $+3[21^4]+2[2^2]+2[2^21]+2[2^21^2]+[2^21^3]+3[2^3]+2[2^31]+[2^4]$ |
| 122 760 | $\{21^2\}$ | $[0]+4[1]+7[1^2]+8[1^3]+8[1^4]+4[1^5]+[2]+4[21]+7[21^2]+6[21^3]$ $+2[21^4]+[2^2]+4[2^21]+5[2^21^2]+2[2^21^3]+[2^3]+2[2^31]+[2^31^2]$ |
| 35 960 | $\{1^4\}$ | $[0]+[1^2]+2[1^3]+2[1^4]+[1^5]+[2]+2[21]+[21^2]+2[21^3]+2[21^4]+3[2^2]$ $+2[2^21]+[2^21^2]+[2^21^3]+[2^3]$ |

**Table 3.** Plethysms $\Delta''\otimes\{\mu\}$ for $SO_{10}$.

$$\Delta\otimes\{2\}=[1^5]_+-[1^4]+[1^3]-[1^2]+[1]-[0]$$

$$\Delta''\otimes\{3\}=\Delta''([1^5]_+-[1^2])$$

$$\Delta''\otimes\{21\}=-\Delta''([1^4]-[1^3]-[1]+[0])$$

$$\Delta''\otimes\{4\}=(2[1^2]+2[1^4]+[2]+2[21^2]+3[21^4]_++[2^2]+[2^21^2]+[2^31^2]_++[2^5]_+)-([1]+2[1^3]+3[1^5]_+ +2[21]+2[21^3]+[2^21]+2[2^21^3]_++[2^41]_+)$$

$$\Delta''\otimes\{31\}=(3[0]+7[1^2]+9[1^4]+2[2]+5[21^2]+4[21^4]_++2[21^4]_-+4[2^21^2]+2[2^2]+[2^3]+2[2^31^2]_++[2^4]) -(5[1]+8[1^3]-6[1^5]_++3[1^5]_-+4[21]+6[21^3]+3[2^21]+3[2^21^3]_+ +[2^21^3]_-+2[2^31]+[2^41]_+)$$

$$\Delta''\otimes\{2^2\}=(2[0]+4[1^2]+6[1^4]+2[2]+4[21^2]+2[21^4]_++2[21^4]_-+2[2^21^2]+2[2^3]+[2^31^2]_++[2^31^2]_-) -(4[1]+6[1^3]+3[1^5]_++3[1^5]_-+2[21]+4[21^3]+2[2^21]+[2^21^3]_+ +[2^21^3]_-+2[2^31])$$

rank three partitions we note, as a check, that from (7.1)

$$\Delta''^3 = \Delta'' \otimes (\{3\}+2\{21\}+\{1^3\}) = \Delta''\Delta''^2 \qquad (10.1)$$

with $\Delta''^2$ given by (5.2*b*).

With the above results established, it becomes possible to evaluate plethysms of the form $\Delta_\pm \otimes \{\mu\}$. This is equivalent to determining the branching rules for $U_{16} \downarrow SO_{10}$, and the results for partitions of four or less are given in table 4. For the symmetric *S*-functions $\{m\}$ of $U_{16}$ it appears that

$$U_{16} \downarrow SO_{10} \qquad \{m\} \downarrow \sum_{x=0}^{[m/2]} \left[\frac{m}{2}, \left(\frac{m}{2}-x\right)^4\right]. \qquad (10.2)$$

This rule, whilst dimensionally correct, must remain merely a conjecture, since we have been unable to prove that in the resulting terms $[\lambda]$ we necessarily have $\lambda_5 \geq 0$.

The plethysms $\square \otimes \{\mu\} = [1^5] \otimes \{\mu\}$ associated with the branching for $U_{252} \downarrow O_{10}$ are listed for partitions $\mu$ of rank three or less in table 5. They may then be determined

**Table 4.** $U_{16} \downarrow SO_{10}$ reductions corresponding to the plethysms $\Delta_+ \otimes \{\mu\}$.

| $D_{\{\mu\}}$ | $U_{16}$ | $SO_{10}$ |
|---|---|---|
| 16 | $\{1\}$ | $[\Delta; 0]_+$ |
| 136 | $\{2\}$ | $[1]+[1^5]_+$ |
| 120 | $\{1^2\}$ | $[1^3]$ |
| 816 | $\{3\}$ | $[\Delta; 1]_+ + [\Delta; 1^5]_+$ |
| 1360 | $\{21\}$ | $[\Delta; 0]_- + [\Delta; 1]_+ + [\Delta; 1^3]_+$ |
| 560 | $\{1^3\}$ | $[\Delta; 1^2]_-$ |
| 3876 | $\{4\}$ | $[2]+[21^4]_+ + [2^5]_+$ |
| 9180 | $\{31\}$ | $[1^2]+[1^4]+[21^2]+[21^4]_+ + [2^31^2]_+$ |
| 5168 | $\{2^2\}$ | $[0]+[1^4]+[2]+[21^4]_+ + [2^3]$ |
| 7140 | $\{21^2\}$ | $[1^2]+[1^4]+[21^2]+[2^21^2]$ |
| 1820 | $\{1^4\}$ | $[21^4]_- + [2^2]$ |

**Table 5.** $U_{252} \downarrow O_{10}$ reductions corresponding to the plethysms $[1^5] \otimes \{\mu\}$.

| $D_{\{\mu\}}$ | $U_{252}$ | $O_{10}$ |
|---|---|---|
| 252 | $\{1\}$ | $[1^5]$ |
| 31 878 | $\{2\}$ | $[2^5]+[2^4]+2[2^3]+[2^21^2]+[2^2]+[21^4]+2[2]+[1^4]+[1^2]+[0]$ |
| 31 626 | $\{1^2\}$ | $[2^4]+[2^31^2]+[2^21^2]+[2^2]+2[21^2]+[1^4]+[1^2]+[0]$ |
| 2699 004 | $\{3\}$ | $[3^5]+[3^41]+2[3^31^2]+[3^22^21]+2[3^221]+3[3^21^3]+[32^4]+2[32^3]+2[32^21^2]$ $+4[32^2]+4[321^2]+2[32]+4[31^4]+2[3]+2[2^41]+6[2^31]+3[2^21^3]$ $+6[2^21]+8[21^3]+2[21]+5[1^5]+2[1^3]+2[1]$ |
| 5334 252 | $\{21\}$ | $[3^41]+[3^32^2]+2[3^32]+2[3^31^2]+2[3^32^21]+6[3^221]+3[3^21^3]+2[3^21]$ $+[32^4]+4[32^3]+5[32^21^2]+4[32^2]+10[321^2]+2[32]+5[31^4]$ $+4[31^2]+4[2^41]+10[2^31]+9[2^21^3]+10[2^21]+16[21^3]$ $+6[21]+7[1^5]+6[1^3]+2[1]$ |
| 2635 500 | $\{1^3\}$ | $2[3^32]+2[3^3]+2[3^22^21]+[3^21^3]+2[3^221]+2[3^21]+2[32^3]+3[32^21^2]$ $+6[321^2]+[31^4]+4[31^2]+[2^41]+6[2^31]+5[2^21^3]+4[2^21]$ $+8[21^3]+2[21]+2[1^5]+6[1^3]$ |

via (7.18), from the tabulations of the plethysms $\{1^5\}\otimes\{\mu\}$ given elsewhere (Butler and Wybourne 1971).

The plethysms $\Box''\otimes\{\mu\}$ are somewhat more tedious to evaluate, but this may be done through the use of (7.17). We list the independent plethysms for partitions $\mu$ of weight three or less in table 6. A complete list may then be obtained through the use of (7.14) once again.

Finally, the plethysms $[1^5]_+\otimes\{\mu\}$ associated with the restriction $U_{126}\downarrow SO_{10}$ are listed in table 7 for partitions $\mu$ of three or less.

**Table 6.** Plethysms $\Box''\otimes\{\mu\}$ for $SO_{10}$.

$\Box''\otimes\{2\}=([2]+[21^2]+[2^3]+[21^4]_++[2^31^2]_-+[2^5]_+)-([0]+[1^2]+[1^4]+[2^2]+[2^21^2]+[2^4])$

$\Box''\otimes\{3\}=\Box''(([1^2]+[21^4]_++[2^3]+[2^5]_+)-([0]+[1^4]+[2^2]+[2^31^2]_+))$

$\Box''\otimes\{21\}=\Box''(([2]+[21^2]+[2^31^2]_++[2^31^2]_-)-(2[1^2]+[2^21^2]+[2^4]))$

**Table 7.** $U_{126}\downarrow SO_{10}$ reductions corresponding to the plethysms $[1^5]_+\otimes\{\mu\}$.

| $D_{\{\mu\}}$ | $U_{126}$ | $SO_{10}$ |
|---|---|---|
| 126 | {1} | $[1^5]_+$ |
| 8 001 | {2} | $[2^5]_++[2^3]+[21^4]_++[2]$ |
| 7 875 | $\{1^2\}$ | $[2^31^2]_++[21^2]$ |
| 341 376 | {3} | $[3^5]_++[3^31^2]_++[3^21^3]_-+[32^4]_++[32^21^2]_++[32^2]+2[31^4]_++[3]+[2^31]+[2^21]+[21^3]+[1^5]_-$ |
| 66 675 | {21} | $[3^32^2]_++[3^21^2]_++[3^221]+[32^4]_++2[32^21^2]_++[32^2]+[321^2]+2[31^4]_++[31^2]+[2^41]_++[2^31]+[2^21^3]_++[2^21^3]_-+[2^21]+2[21^3]+[21]+[1^5]_-$ |
| 325 500 | $\{1^3\}$ | $[3^3]+[3^22^21]_++[32^21^2]_++[321^2]+[31^2]+[2^31]+[2^21^3]_++[21^3]+[1^3]$ |

## 11. $SO_{10}$ invariant polynomials and Higgs potentials

The construction of a Higgs potential that is both a symmetric quartic polynomial in a set of Higgs scalar fields and an invariant with respect to some symmetry group G such as $SO_{10}$ has been discussed elsewhere (Wybourne 1980). If the Higgs scalar fields $\phi$ form the basis of a unitary representation $\lambda$ of G, which is in general reducible, then the number of independent parameters in the Higgs potential is the number of times the trivial, identity representation appears in the plethysm

$$\lambda\otimes(\{2\}+\{3\}+\{4\}). \tag{11.1}$$

In grand unified models based on $SO_{10}$, it is of interest to consider the case corresponding to the set of Higgs scalar fields

$$\phi=\{\phi_a,\phi_+,\phi_-\}. \tag{11.2}$$

The fields $\phi_a$ transform as the basis states of the adjoint irrep $[1^2]$ of dimension $\underline{45}$, whilst $\phi_\pm$ transform as basis states of the irreps $[1^5]_\pm$ of dimension $\underline{126}$, so that

$$\lambda = [1^2]+[1^5]_+ +[1^5]_-. \tag{11.3}$$

It should be noted that the irreps $[\lambda]$ of $SO_{10}$ with $\lambda_5 = 0$ are real and orthogonal, so that

$$[\lambda][\lambda] \supset [0], \qquad [\lambda][\mu] \not\supset [0] \qquad \text{if } \mu \neq \lambda, \tag{11.4a}$$

$$[\lambda]\otimes\{2\} \supset [0], \qquad [\lambda]\otimes\{1^2\} \not\supset [0], \tag{11.4b}$$

whilst the irreps $[\lambda]_+$ and $[\lambda]_-$ with $\lambda_5 \neq 0$ are mutually complex conjugate, neither orthogonal nor symplectic, so that

$$[\lambda]_\pm[\lambda]_\pm \not\supset [0], \qquad [\lambda]_\pm[\mu]_\pm \not\supset [0] \qquad \text{if } \mu \neq \lambda, \tag{11.5a}$$

$$[\lambda]_\pm[\lambda]_\mp \supset [0], \qquad [\lambda]_\pm[\mu]_\mp \not\supset [0] \qquad \text{if } \mu \neq \lambda, \tag{11.5b}$$

$$[\lambda]_\pm \otimes\{2\} \not\supset [0], \qquad [\lambda]_\pm \otimes\{1^2\} \not\supset [0]. \tag{11.5c}$$

These results are exemplified, in part, by the formulae

$$[1^2]\otimes\{2\} = [2^2]+[2]+[1^4]+[0], \tag{11.6a}$$

$$[1^5]_\pm \otimes\{2\} = [2^5]_+ +[2^3]+[21^4]_+ +[2], \tag{11.6b}$$

$$[1^5]_+[1^5]_- = [2^4]+[2^21^2]+[2^2]+[1^4]+[1^2]+[0]. \tag{11.6c}$$

Turning to the question of second-order invariants, the expansion

$$\begin{aligned}([1^2]+[1^5]_+ +[1^5]_-)\otimes\{2\} &= [1^2]\otimes\{2\}+[1^5]_+\otimes\{2\} \\ &\quad +[1^5]_-\otimes\{2\}+[1^2][1^5]_+ +[1^2][1^5]_- +[1^5]_+[1^5]_-\end{aligned} \tag{11.7}$$

is such that from (11.6) it is clear that

$$([1^2]+[1^5]_+ +[1^5]_-)\otimes\{2\} \supset 2[0], \tag{11.8}$$

with the two invariants, arising from $[1^2]\otimes\{2\}$ and $[1^5]_+[1^5]_-$, conveniently specified in the notation of (11.2) by

$$[\phi_a^2]_{[0]} \qquad \text{and} \qquad [\phi_+\phi_-]_{[0]}. \tag{11.9}$$

The first of these is just the usual second-order Casimir invariant.

In the third-order case

$$\begin{aligned}([1^2]+[1^5]_+ +[1^5]_-)\otimes\{3\} &= [1^2]\otimes\{3\}+[1^5]_+\otimes\{3\}+[1^5]_-\otimes\{3\} \\ &\quad +([1^2]\otimes\{2\})([1^5]_+ +[1^5]_-)+([1^5]_+\otimes\{2\})([1^2]+[1^5]_-) \\ &\quad +([1^5]_-\otimes\{2\})([1^2]+[1^5]_+)+[1^2][1^5]_+[1^5]_-\end{aligned} \tag{11.10}$$

whilst

$$[1^2]\otimes\{3\} = [3^2]+[31]+[2^21^2]+[21^2]+[1^4]+2[1^2], \tag{11.11a}$$

$$\begin{aligned}[1^5]_\pm\otimes\{3\} &= [3^5]_\pm +[3^31^2]_\pm +[3^21^2]_\mp +[32^4]_\pm +[32^21^2]_\pm \\ &\quad +[32^2]+2[31^4]_\pm +[3]+[2^31]+[2^21]+[21^3]+[1^5]_\mp.\end{aligned} \tag{11.11b}$$

Hence using the general results (11.4) and (11.5), along with (11.6) and (11.11), gives

$$([1^2]+[1^5]_+ +[1^5]_-)\otimes\{3\} \supset [0] \tag{11.12}$$

with the single invariant, arising from $[1^2]\,[1^5]_+\,[1^5]_-$, specified by

$$[\phi_a\phi_+\phi_-]_{[0]}. \tag{11.13}$$

In the fourth-order case

$$\begin{aligned}([1^2]+[1^5]_++[1^5]_-)\otimes\{4\}=&[1^2]\otimes\{4\}+[1^5]_+\otimes\{4\}+[1^5]_-\otimes\{4\}\\ &+([1^2]\otimes\{3\})([1^5]_++[1^5]_-)+([1^5]_+\otimes\{3\})([1^2]+[1^5]_-)\\ &+([1^5]_-\otimes\{3\})([1^2]+[1^5]_+)+([1^2]\otimes\{2\})([1^5]_+\otimes\{2\})\\ &+([1^2]\otimes\{2\})([1^5]_-\otimes\{2\})+([1^5]_+\otimes\{2\})([1^5]_-\otimes\{2\})\\ &+([1^2]\otimes\{2\})[1^5]_+[1^5]_-+([1^5]_+\otimes\{2\})[1^2][1^5]_-\\ &+([1^5]_-\otimes\{2\})[1^2][1^5]_+.\end{aligned} \tag{11.14}$$

Taking the terms one by one: the invariants in $[1^2]\otimes\{4\}$ can be found by using (7.8*a*) and noting that from (11.11*a*) and (11.4*a*)

$$([1^2]\otimes\{3\})[1^2]\supset 2[0] \tag{11.15a}$$

whilst from (11.6*a*) and (11.4*b*)

$$([1^2]\otimes\{2\})\otimes\{1^2\}=([2^2]+[2]+[1^4]+[0])\otimes\{1^2\}\not\supset[0]. \tag{11.15b}$$

The last step depends upon the use of (7.3*d*), which implies that

$$(A+B+C+\ldots)\otimes\{1^2\}=A\otimes\{1^2\}+B\otimes\{1^2\}+\ldots+AB+AC+\ldots+BC+\ldots. \tag{11.16}$$

Thus

$$[1^2]\otimes\{4\}\supset 2[0]. \tag{11.17}$$

Similarly from (11.11*b*) and (11.4)

$$([1^5]_\pm\otimes\{3\})[1^5]_\pm\supset[0] \tag{11.18a}$$

whilst from (11.6*b*) and (11.4)

$$([1^5]_\pm\otimes\{2\})\otimes\{1^2\}\not\supset[0] \tag{11.18b}$$

so that from (7.8*a*)

$$[1^5]_\pm\otimes\{4\}\supset[0]. \tag{11.19}$$

The terms $([1^2]\otimes\{3\})[1^5]_\pm$ and $([1^5]_\pm\otimes\{3\})([1^2]+[1^5]_\mp)$ do not contain [0], as may be seen from (11.11).

From (4.11*c*) and (4.14)

$$[1^2][1^5]_\pm=[2^21^3]_\pm+[1^5]_\pm+[21^3]+[1^3] \tag{11.20}$$

so that comparison with (11.6*b*) indicates that $([1^5]_\pm\otimes\{2\})[1^2][1^5]_\mp$ also do not contain [0].

All the remaining terms contribute invariants:

$$([1^2]\otimes\{2\})([1^5]_\pm\otimes\{2\})\supset[0], \tag{11.21a}$$

$$([1^5]_+\otimes\{2\})([1^5]_-\otimes\{2\})\supset 4[0], \tag{11.21b}$$

$$([1^2]\otimes\{2\})[1^5]_+[1^5]_-\supset 3[0]. \tag{11.21c}$$

Thus (11.17), (11.19) and (11.21) indicate that

$$([1^2]+[1^5]_+ + [1^5]_-)\otimes\{4\} \supset 15[0] \tag{11.22}$$

with the 15 invariants given explicitly by

$$[\phi_a^4]_{[0]}, \qquad ([\phi_a^2]_{[0]})^2, \tag{11.23$a$}$$

$$[\phi_+^4]_{[0]}, \qquad [\phi_-^4]_{[0]}, \tag{11.23$b$}$$

$$([\phi_a^2]_{[2]}[\phi_+^2]_{[2]}), \qquad ([\phi_a^2]_{[2]}[\phi_-^2]_{[2]}), \tag{11.23$c$}$$

$$([\phi_+^2]_{[2^5]_+}[\phi_-^2]_{[2^5]_+}), \qquad ([\phi_+^2]_{[2^3]}[\phi_-^2]_{[2^3]}),$$

$$([\phi_+^2]_{[21^4]_+}[\phi_-^2]_{[21^4]_-}), \qquad ([\phi_+^2]_{[2]}[\phi_-^2]_{[2]}), \tag{11.23$d$}$$

$$([\phi_a^2]_{[2^2]}[\phi_+\phi_-]_{[2^2]}), \qquad ([\phi_a^2]_{[1^4]}[\phi_+\phi_-]_{[1^4]}), \qquad ([\phi_a^2]_{[2]}[\phi_+\phi_-]_{[2]}). \tag{11.23$e$}$$

These are not all independent of the second-order invariants (11.9), one linear combination of the terms in (11.23$a$), (11.23$d$) and (11.23$e$) being given by

$$([\phi_a^2]_{[0]})^2, \qquad ([\phi_+\phi_-]_{[0]})^2 \qquad \text{and} \qquad ([\phi_a^2]_{[0]}[\phi_+\phi_-]_{[0]})$$

respectively.

This leaves a total of two independent second-order invariants, one independent third-order invariant and ten independent fourth-order invariants.

## 12. Conclusions

The results obtained in this paper permit the unambiguous evaluation of all possible Kronecker products of irreps (tensor and spinor) of $O_n$ and $SO_n$. A complete prescription for resolving up to fourth powers of any irrep of $O_n$ and $SO_n$ has been found. These results should be relevant to the development of grand unified theories. The resolution of the Kronecker squares and cubes finds important applications in the calculation of symmetrised *nj* symbols.

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